LAGRANGIAN SUBBUNDLES AND CODIMENSION 3 SUBCANONICAL SUBSCHEMES

DAVID EISENBUD, SORIN POPESCU, AND CHARLES WALTER

Abstract. We show that a Gorenstein subcanonical codimension 3 subscheme $Z \subset X = \mathbb{P}^N$, $N \geq 4$, can be realized as the locus along which two Lagrangian subbundles of a twisted orthogonal bundle meet degenerately, and conversely. We extend this result to singular Z and all quasiprojective ambient schemes X under the necessary hypothesis that Z is *strongly subcanonical* in a sense defined below. A central point is that a pair of Lagrangian subbundles can be transformed locally into an alternating map. In the local case our structure theorem reduces to that of Buchsbaum-Eisenbud [6] and says that Z is Pfaffian.

We also prove codimension one symmetric and skew-symmetric analogues of our structure theorems.

Smooth subvarieties of small codimension $Z \subset X = \mathbb{P}^N$ have been extensively studied in recent years, especially in relation to Hartshorne's conjecture that a smooth subvariety of sufficiently small codimension in $\mathbb{P}^N$ is a complete intersection. Although the conjecture remains open, any smooth subvariety Z of small codimension in $\mathbb{P}^N$ is known, by a theorem of Barth, Larsen, and Lefschetz, to have the weaker property that it is *subcanonical* in the sense that its canonical class is a multiple of its hyperplane class.

More generally, a subscheme Z of a nonsingular Noetherian scheme X is said to be subcanonical if Z is Gorenstein and its canonical bundle is the restriction of a bundle on X. There is a natural generalization to an arbitrary (possibly singular) scheme X (see below).

In this paper we give a structure theorem for subcanonical subschemes of codimension 3 in $\mathbb{P}^N$ and generalize it to subcanonical subschemes of codimension 3 in an arbitrary quasiprojective scheme X satisfying a mild extra cohomological condition (strongly subcanonical subschemes). The construction works even without the quasiprojective hypothesis.

There are well known theorems describing the local structure of Gorenstein subschemes of nonsingular Noetherian schemes in codimensions ≤ 3. In codimensions 1 and 2 all Gorenstein subschemes are locally complete intersections. These results have been globalized: If X is nonsingular, any $Z \subset X$ of codimension 1 is the zero locus of a section of a line bundle, while a subcanonical $Z \subset X$ of codimension 2 is the zero locus of a section of

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a rank 2 vector bundle if a certain obstruction in cohomology vanishes (as explained below). In both cases $\mathcal{O}_Z$ has a symmetric resolution by locally free $\mathcal{O}_X$-modules.

In codimension 3 both the local and the global cases become more complicated. Locally, a Gorenstein subscheme of codimension 3 need not be a locally complete intersection. Rather, Buchsbaum and Eisenbud [6] showed that such a subscheme is cut out locally by the submaximal Pfaffians of an alternating matrix appearing in a minimal free resolution. Okonek [28] asked whether this local result could be generalized to show that codimension 3 subcanonical schemes are cut out by the Pfaffians of an alternating map of vector bundles. Walter [34] gave a positive answer to Okonek's question in $\mathbb{P}^n$ under a mild additional hypothesis, but left open the question of whether this hypothesis is always satisfied.

In our paper [13] we will show that *not every subcanonical subscheme of codimension* 3 *in* $\mathbb{P}^n$ *is Pfaffian,* settling Okonek's question negatively. But in the present paper we show that a different way of looking at the Pfaffian construction does generalize, and gives the desired structure theorem for all subcanonical subschemes of codimension 3. (The question as to which subschemes are Pfaffian can be answered in the derived Witt group of Balmer [2]; see Walter [35].)

In this paper a closed subscheme $Z \subset X$ of a Noetherian scheme is called *subcanonical of codimension* d if it satisfies two conditions:

A) The subscheme Z is *relatively Cohen-Macaulay of codimension* d in X, i.e. $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_X) = 0$ for all $i \neq d$, and

B) There exists a line bundle L on X such that the relative canonical sheaf $\omega_{Z/X} := \mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_X)$ is isomorphic to the restriction of L^{-1} to Z.

These conditions are not enough for the Serre correspondence in codimension 2, nor for our structure theorem in codimension 3. Condition B asserts the existence of an isomorphism

$$\eta : \mathcal{O}_Z \xrightarrow{\sim} \omega_{Z/X}(L) = \mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{O}_Z, L)$$

which one can think of as an $\eta \in H^0(\mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{O}_Z, L)) = \operatorname{Ext}^d_{\mathcal{O}_X}(\mathcal{O}_Z, L)$. In the Yoneda Ext, this η defines a class of "resolutions of $\mathcal{O}_Z$ by coherent sheaves"

$$0 \to L \to \mathcal{F}_{d-1} \to \cdots \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{O}_Z \to 0.$$

In our structure theorem we require such resolutions with $\mathcal{F}_0, \dots, \mathcal{F}_{d-1}$ locally free. Thus we will need the condition

C) The $\mathcal{O}_X$-module $\mathcal{O}_Z$ is of finite local projective dimension (necessarily equal to the codimension d).

This condition holds automatically if the ambient scheme X is nonsingular.

We also need $\mathcal{F}_0 = \mathcal{O}_X$, which means that we want $\eta \in \operatorname{Ext}^d_{\mathcal{O}_X}(\mathcal{O}_Z, L)$ to lift to $\operatorname{Ext}^{d-1}_{\mathcal{O}_X}(\mathcal{I}_Z, L)$. Since these two groups are joined by a map in the long exact sequence obtained by applying $\operatorname{Ext}^*_{\mathcal{O}_X}(-, L)$ to the short exact

sequence $0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$, we see that the lifting exists if and only if $\eta \in \mathrm{Ext}^d_{\mathcal{O}_X}(\mathcal{O}_Z, L)$ goes to 0 in $\mathrm{Ext}^d_{\mathcal{O}_X}(\mathcal{O}_X, L) \cong H^d(X, L)$. We are thus led to the condition:

D) The isomorphism class $\eta \in \mathrm{Ext}^d_{\mathcal{O}_X}(\mathcal{O}_Z, L)$ of (2) goes to zero under the map

$$\mathrm{Ext}^d_{\mathcal{O}_X}(\mathcal{O}_Z, L) \to \mathrm{Ext}^d_{\mathcal{O}_X}(\mathcal{O}_X, L) = H^d(X, L)$$

induced by the surjection $\mathcal{O}_X \to \mathcal{O}_Z$.

Condition D holds automatically if $H^d(X, L) = 0$. This is the case if $X = \mathbb{P}^n$ with $n \geq d+1$, or if X is an affine scheme. In addition, if the ambient scheme X is a Gorenstein variety over a field k, then condition D can be put into a dual form which looks more natural. For in that case $Z \subset X$ is subcanonical (of dimension r) if and only if it is Cohen-Macaulay and there exists a line bundle M on X such that $\omega_Z \cong M|_Z$, and condition D holds if and only if the following composite map vanishes

$$H^r(X, M) \xrightarrow{\text{rest}} H^r(Z, M|_Z) \underset{\cong}{\xrightarrow{\eta}} H^r(Z, \omega_Z) \xrightarrow{\text{tr}} k. \tag{1}$$

In these terms we may give the central definition of this paper:

Definition 0.1. A subscheme $Z \subset X$ is *strongly subcanonical* if it satisfies conditions A-D.

The Serre construction shows that a subscheme of codimension 2 is the zero locus of a rank 2 vector bundle if and only if it is strongly subcanonical (Griffiths-Harris [18] Proposition 1.33, Vogelaar [33] Theorem 2.1, and Bănică-Putinar [3] §2.1 state variants of condition D explicitly).

Our main results show that a codimension 3 subscheme Z of a quasiprojective scheme X is strongly subcanonical if and only if it can be expressed as an appropriate "Lagrangian degeneracy scheme," defined as follows: Let $\mathcal{V}$ be a vector bundle on X of even rank $2n$ equipped with a nonsingular quadratic form q with values in a line bundle L. Let $\mathcal{E}$ and $\mathcal{F}$ be a pair of Lagrangian subbundles of $(\mathcal{V}, q)$ (i.e. totally isotropic subbundles of rank n). It is then well known that $\dim[\mathcal{E}(x) \cap \mathcal{F}(x)]$ is locally constant modulo 2.

Now suppose that m is an integer such that $\dim[\mathcal{E}(x) \cap \mathcal{F}(x)] \equiv m \pmod 2$ for all $x \in X$. Then there is a degeneracy locus which as a set is given by

$$Z_m(\mathcal{E}, \mathcal{F})_{\mathrm{red}} := \left\{ x \in X \mid \dim_{k(x)}[\mathcal{E}(x) \cap \mathcal{F}(x)] \geq m \right\}.$$

In §2 we will define a scheme structure on this set in roughly the following manner. Using the data $\mathcal{E}, \mathcal{F} \subset (\mathcal{V}, q)$ one defines a composite map

$$\lambda : \quad \mathcal{E} \to \mathcal{V} \cong \mathcal{V}^*(L) \to \mathcal{F}^*(L)$$

such that $\ker(\lambda(x)) = \mathcal{E}(x) \cap \mathcal{F}(x)$ for all $x \in X$. Even if $\mathcal{E} \cong \mathcal{F}$ the map λ may not be alternating, but (perhaps after modifying λ slightly to make it have even rank everywhere) we will show that it is possible to find local bases in which the matrix of λ is alternating (Proposition 2.3). Although these alternating matrices do not glue together, they are sufficiently compatible

that we can define $Z_m(\mathcal{E},\mathcal{F})$ as the locus defined by their Pfaffians of order $\mathrm{rk}(\mathcal{E})-m+2$. This scheme structure is natural in the sense that in a suitably generic setting it is reduced, and it is stable under base change.

The following structure theorem for strongly subcanonical codimension 3 subschemes collects the main results of this paper:

Theorem 0.2. *Let X be a quasiprojective scheme over a Noetherian ring, and let $Z \subset X$ be a closed subscheme of grade 3. The following conditions are equivalent*:

(a) *$Z \subset X$ is strongly subcanonical.*

(b) *There exists a twisted orthogonal bundle $(\mathcal{V}, q)$ and Lagrangian subbundles $\mathcal{E},\mathcal{F} \subset (\mathcal{V}, q)$, with $\dim_{k(x)}\big[\mathcal{E}(x)\cap\mathcal{F}(x)\big]$ odd for all $x \in X$, such that $Z = Z_3(\mathcal{E},\mathcal{F})$.*

(c) *There exists a vector bundle $\mathcal{F}$, a line bundle L, and a Lagrangian subbundle $\mathcal{E}$ of the hyperbolic bundle $\mathcal{F} \oplus \mathcal{F}^*(L)$, such that the composite map*

$$\lambda : \mathcal{E} \hookrightarrow \mathcal{F} \oplus \mathcal{F}^*(L) \twoheadrightarrow \mathcal{F}^*(L)$$

has kernel of odd rank and such that $Z = Z_3(\mathcal{E},\mathcal{F}^*(L))$.

(d) *Z has symmetrically quasi-isomorphic locally free resolutions*

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & \mathcal{H} \arrow[r, "\psi"] \arrow[d, "\phi"'] & \mathcal{G} \arrow[r] \arrow[d, "\phi^*"] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & \mathcal{O}_Z \arrow[r] \arrow[d, "\cong"', "\eta"] & 0 \\
0 \arrow[r] & L \arrow[r] & \mathcal{G}^*(L) \arrow[r, "-\psi^*"] & \mathcal{H}^*(L) \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{E}xt^3_{\mathcal{O}_X}(\mathcal{O}_Z, L) \arrow[r] & 0
\end{tikzcd}$$

where L is a line bundle on X, and $\phi^*\psi : \mathcal{E} \to \mathcal{E}^*(L)$ is an alternating map.

The structure theorem will be proved in several parts: see Theorems 3.1, 4.1, and 6.1.

One way to look at the structure theorem is as follows. The existence of the symmetric isomorphism

$$\eta : \mathcal{O}_Z \xrightarrow{\sim} \mathcal{E}xt^3_{\mathcal{O}_X}(\mathcal{O}_Z, L) \tag{2}$$

means that there should be a symmetric isomorphism in the derived category from the locally free resolution of $\mathcal{O}_Z$

$$0 \to L \xrightarrow{q} \mathcal{E} \xrightarrow{\psi} \mathcal{G} \xrightarrow{p} \mathcal{O}_X \to \mathcal{O}_Z \to 0 \tag{3}$$

into its twisted shifted dual. In general, morphisms in the derived category are complicated objects involving homotopy classes of maps and a calculus of fractions. Nevertheless, in Theorem 6.1 we show that there exist such locally free resolutions of $\mathcal{O}_Z$—which depend on the choice of η—for which the symmetric isomorphism in the derived category is induced by a symmetric chain map which is a quasi-isomorphism and therefore becomes an isomorphism in the derived category. Okonek's Pfaffian subschemes correspond to situations where this quasi-isomorphism is an isomorphism.

The philosophy that (skew)-symmetric sheaves should have locally free resolutions that are (skew)-symmetric up to quasi-isomorphism is also pursued in our paper [14] and in Walter [35]. The former deals primarily with methods for constructing explicit locally free resolutions for (skew)-symmetric sheaves on $\mathbb{P}^n$. The latter studies the obstructions (in Balmer's derived Witt groups [2]) to the existence of a genuinely (skew)-symmetric resolution.

The results of this paper give a full characterization of codimension 3 subcanonical subschemes. In [13] we use this machinery to construct various geometric examples of subcanonical subschemes of codimension 3 which are not Pfaffian.

Porteous-type formulas for the fundamental classes of degeneracy loci for skew-symmetric maps $\phi : \mathcal{E} \to \mathcal{E}^*(L)$ were found by Harris-Tu [20], Józefiak-Lascoux-Pragacz [22], and Pragacz [31]. Harris asked for similar formulas for degeneracy loci related to pairs of Lagrangian subbundles, and they were provided by Fulton [15] [16] and Pragacz-Ratajski [32] (see Fulton-Pragacz [17] for more details). (A scheme structure on these degeneracy loci and their generalizations with isotropic flag conditions can be defined in a manner similar to (12) below.)

Fulton and Pragacz ([17] §9.4) also ask whether one can find "natural" resolutions for the structure sheaves of these kinds of symmetric and skew-symmetric degeneracy loci. From such a resolution one can read off formulas in $K_0(X)$. Theorem 3.1 provides an explicit answer in one simple case.

Structure of the paper. In Sections 1 and 2 we review basic facts about Lagrangian subbundles of twisted orthogonal bundles and define the scheme structure on the degeneracy loci $Z_m(\mathcal{E}, \mathcal{F})$. In Section 3 we prove that Lagrangian degeneracy loci of codimension 3 are strongly subcanonical (Theorem 3.1). In Section 4 we discuss "split" Lagrangian degeneracy loci (Theorem 4.1) which are often more practical for constructing codimension 3 subcanonical subschemes. The computation of local equations for these degeneracy loci is discussed in Section 5.

In Section 6 we complete the proof of the structure Theorem 0.2 by showing that strongly subcanonical subschemes of codimension 3 are split Lagrangian degeneracy loci (Theorem 6.1). Sections 7 and 8 discuss at length various examples of codimension 3 subcanonical subschemes, particularly the case of points in $\mathbb{P}^3$. Further examples can be found in our paper [13].

Finally, in Section 9 we prove codimension one symmetric and skew-symmetric analogues of all previous results. In particular we state Casnati-Catanese's structural result ([7] Remark 2.2) and give an example of a self-linked threefold of degree 18 in $\mathbb{P}^5$ which does not have a symmetric resolution because the parity condition fails.

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1. Quadratic forms on vector bundles

In this section we recall the basic definitions of twisted orthogonal bundles and of Lagrangian subbundles. The definitions and results can be found in many standard references such as Fulton-Pragacz [17] Chap. 6, Knus [24], and Mukai [25] §1.

Quadratic forms. Suppose that V is a finite-dimensional vector space over a field k. (We impose no restrictions on k; it may have characteristic 2, and need not be algebraically closed.) A quadratic form on V is a homogeneous quadratic polynomial in the linear forms on V, i.e. a member $q \in S^2(V^*)$. The symmetric bilinear form $b : V \times V \to k$ associated to q is given by the formula

$$b(x, y) := q(x + y) - q(x) - q(y). \tag{4}$$

The quadratic form q is *nondegenerate* if b is a perfect pairing.

Now suppose that $\mathcal{V}$ is a locally free sheaf of constant finite rank over a scheme X. A quadratic form on $\mathcal{V}$ with values in a line bundle L is a global section q of $S^2(\mathcal{V}^*) \otimes L$. Such a quadratic form is *nonsingular* if the induced symmetric bilinear form is a perfect pairing. Equivalently a quadratic form q on $\mathcal{V}$ is nonsingular if for each point $x \in X$ the induced quadratic form $q(x)$ on the fiber vector space $\mathcal{V}(x)$ is nondegenerate. A *twisted orthogonal bundle* on X is a vector bundle $\mathcal{V}$ equipped with a nonsingular quadratic form q with values in some line bundle L.

Lagrangian subbundles. If V is a vector space of even dimension $2n$ equipped with a nondegenerate quadratic form, then a *Lagrangian subspace* $E \subset (V, q)$ is a subspace of V of dimension n such that $q|_E \equiv 0$. If the characteristic is $\neq 2$, then $E \subset (V, q)$ is Lagrangian if and only if $E = E^\perp := \{x \in V \mid b(x, y) = 0 \text{ for all } y \in E\}$. But in characteristic 2 this condition is necessary but not sufficient for q to vanish on E, i.e. for E to be Lagrangian.

Similarly, a *Lagrangian subbundle* $\mathcal{E} \subset (\mathcal{V}, q)$ of a twisted orthogonal bundle of even rank $2n$ is a subbundle (with locally free quotient sheaf) of rank n such that $q|_{\mathcal{E}} \equiv 0$.

The following result is well known (cf. Bourbaki [5] §6 ex. 18(d), Mumford [26], Mukai [25] Proposition 1.6).

Proposition 1.1. *If $\mathcal{E}$ and $\mathcal{F}$ are Lagrangian subbundles of a twisted orthogonal bundle over a scheme X, then the function on X given by $x \mapsto \dim_{k(x)}\big[\mathcal{E}(x) \cap \mathcal{F}(x)\big]$ is locally constant modulo 2.*

Hyperbolic bundles. If $\mathcal{F}$ is any vector bundle of constant rank, and L is any line bundle, then then $\mathcal{F} \oplus \mathcal{F}^*(L)$ may be endowed with the *hyperbolic quadratic form* $q_h(e \oplus \alpha) := \alpha(e)$ with values in L. (This q_h is bilinear on $\mathcal{F} \times \mathcal{F}^*(L)$ but quadratic on $\mathcal{F} \oplus \mathcal{F}^*(L)$.) The associated hyperbolic symmetric bilinear form has matrix $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$.

We will use the following notation for graph subbundles. If $\psi : \mathcal{A} \to \mathcal{B}$ and $\alpha : \mathcal{B} \to \mathcal{A}$ are morphisms of vector bundles, then we write

$$\Gamma_\psi := \operatorname{im}\left(\mathcal{A} \xhookrightarrow{\left(\begin{smallmatrix} 1 \\ \psi \end{smallmatrix}\right)} \mathcal{A} \oplus \mathcal{B}\right), \qquad \Gamma_\alpha := \operatorname{im}\left(\mathcal{B} \xhookrightarrow{\left(\begin{smallmatrix} \alpha \\ 1 \end{smallmatrix}\right)} \mathcal{A} \oplus \mathcal{B}\right).$$

These graphs are to be regarded as subbundles of $\mathcal{A} \oplus \mathcal{B}$.

Lemma 1.2. *A subbundle $\mathcal{E} \subset (\mathcal{F} \oplus \mathcal{F}^*(L), q_h)$ is a Lagrangian subbundle complementary to the direct summand $\mathcal{F}^*(L)$ if and only if there is an alternating map $\zeta : \mathcal{F} \to \mathcal{F}^*(L)$ such that $\mathcal{E} = \Gamma_\zeta$.*

Any Lagrangian subbundle of a twisted orthogonal bundle over an **affine** scheme has a Lagrangian complement (cf. [24] Remark I.5.5.4), although this is not always true over a general scheme. However, if a Lagrangian subbundle $\mathcal{F} \subset (\mathcal{V}, q)$ has a Lagrangian complement $\mathcal{M}$, then the symmetric bilinear form induces a natural isomorphism $\mathcal{M} \cong \mathcal{F}^*(L)$. This defines an isometry

$$(\mathcal{V}, q) \xrightarrow[\cong]{\phi_{\mathcal{F},\mathcal{M}}} (\mathcal{F} \oplus \mathcal{F}^*(L), q_h) \tag{5}$$

which is the identity on $\mathcal{F}$ and which identifies the complementary Lagrangian subbundles $\mathcal{F}, \mathcal{M} \subset \mathcal{V}$ with the two direct summands of $\mathcal{F} \oplus \mathcal{F}^*(L)$. The previous lemma has the following corollary.

Corollary 1.3. *If $\mathcal{E}, \mathcal{F} \subset (\mathcal{V}, q)$ are Lagrangian subbundles with a common Lagrangian complement $\mathcal{M}$, then there is an alternating map $\zeta : \mathcal{F} \to \mathcal{F}^*(L)$ such that $\phi_{\mathcal{F},\mathcal{M}}(\mathcal{E}) = \Gamma_\zeta$.*

2. Locally alternating maps and Lagrangian degeneracy loci

In this section we show how to use Corollary 1.3 to define scheme-theoretic degeneracy loci for pairs of Lagrangian subbundles of a twisted orthogonal bundle which generalize the degeneracy loci for alternating maps defined by ideals of Pfaffians. We also show how to turn a pair of Lagrangian subbundles into a locally alternating map. Several steps are required, in order to make sure that common Lagrangian complements exist locally, and to show that our degeneracy loci are independent of the choice of common Lagrangian complement. Our scheme structure defined by local equations coincides with that given by a universal construction in De Concini-Pragacz [10].

Existence of local common Lagrangian complements. The result we need is is standard if the residue field is infinite, but if the residue field is very small, care is required. We are interested in when two Lagrangian subbundles $\mathcal{E}, \mathcal{F}$ of a twisted orthogonal bundle $(\mathcal{V}, q)$ have a common Lagrangian complement locally. If one recalls that an even-dimensional quadratic vector space $(\mathcal{V}(x), q(x))$ has two families of Lagrangian subspaces, and that in order for two Lagrangian subspaces to have a common Lagrangian complement they must lie in the same family, and that this is measured by the dimension of the intersection modulo 2, we see that in order for $\mathcal{E}$ and $\mathcal{F}$ to have a common Lagrangian complement to $\mathcal{E}$ and $\mathcal{F}$, we must have $\dim_{k(x)} [\mathcal{E}(x) \cap \mathcal{F}(x)] \equiv \mathrm{rk}(\mathcal{E}) \pmod 2$. We now show that locally this condition is also sufficient.

Proposition 2.1. *Let $\mathcal{E}, \mathcal{F} \subset (\mathcal{V}, q)$ be Lagrangian subbundles of a twisted orthogonal bundle on a scheme X. Suppose that $\dim_{k(x)}\big[\mathcal{E}(x) \cap \mathcal{F}(x)\big] \equiv \mathrm{rk}(\mathcal{E}) \pmod 2$ for all $x \in X$. Then any $x \in X$ has a neighborhood U over which $\mathcal{E}|_U$ and $\mathcal{F}|_U$ have a common Lagrangian complement $\mathcal{M}_U$.*

Proof. It is easy to see that any common Lagrangian complement at x extends to a common Lagrangian complement in a neighborhood U. The existence at x of such a complement is standard if the residue field is infinite. The following lemma deals with the remaining case:

Lemma 2.2. *Suppose that q is a nondegenerate quadratic form on an even-dimensional vector space V, and that $U, U' \subset (V, q)$ are two Lagrangian subspaces such that $\dim(U \cap U') \equiv \dim(U) \pmod 2$. Then there exists a Lagrangian subspace $L \subset (V, q)$ complementary to U and to U'.*

Proof. Let $K = U \cap U'$. Then $U = U^\perp \subset K^\perp$, and similarly $U' \subset K^\perp$. On dimensional grounds, we must indeed have $U + U' = K^\perp$. As a result U/K and U'/K are complementary Lagrangian subspaces of $K^\perp/K$. Moreover, by hypothesis they are even-dimensional.

Let $f_1, \ldots, f_{2m}$ be a system of vectors in U mapping onto a basis of U/K. Since U/K and U'/K are complementary Lagrangian subspaces of $K^\perp/K$, the symmetric bilinear form b associated to q induces a perfect pairing between them. So there exists a system of vectors $g_1, \ldots, g_{2m}$ in U' such that $b(f_i, g_j) = \delta_{ij}$ for all i, j.

Let N be the subspace spanned by the f_i and g_j. Then $q|_N$ is nondegenerate, so there is an orthogonal direct sum decomposition $V = N \oplus N^\perp$ such that $q|_N$ and $q|_{N^\perp}$ are both nondegenerate. Moreover, $K \subset (N^\perp, q|_{N^\perp})$ is a Lagrangian subspace, for which there exists a complementary Lagrangian subspace P by our previous remarks. Let $p_1, \ldots, p_r$ be a basis of P. One may now check that

$$f_1 + g_2, f_2 - g_1, f_3 + g_4, f_4 - g_3, \ldots, f_{2m-1} + g_{2m}, f_{2m} - g_{2m-1}, p_1, \ldots, p_r$$

form a basis for a Lagrangian subspace $L \subset (V, q)$ complementary to both U and U'. □

The following example shows that Lemma 2.2 does not always extend to three Lagrangian subspaces. Suppose that $k = \mathbb{Z}/2\mathbb{Z}$, that $V = k^4$, and that $q = x_1x_3 + x_2x_4$. Let U, U' and U'' be the Lagrangian subspaces given by $x_1 = x_2 = 0$, by $x_3 = x_4 = 0$, and by $x_1 + x_3 = x_2 + x_4 = 0$, respectively. Each subspace is of dimension 2, and each pair of subspaces has intersection of dimension 0. But there is no Lagrangian subspace of V which is complementary to U, to U' and to U''.

Locally alternating maps. Suppose that $f : \mathcal{E} \hookrightarrow (\mathcal{V}, q)$ and $g : \mathcal{F} \hookrightarrow (\mathcal{V}, q)$ are Lagrangian subbundles of a twisted orthogonal bundle. Consider the composite map

$$\lambda : \quad \mathcal{E} \xrightarrow{f} \mathcal{V} \xrightarrow{\beta} \mathcal{V}^*(L) \xrightarrow{g^*} \mathcal{F}^*(L) \tag{6}$$

where $\beta : \mathcal{V} \xrightarrow{\cong} \mathcal{V}^*(L)$ is the isomorphism induced by the quadratic form q. In the special case of Lemma 1.2, λ is the alternating map $\zeta : \mathcal{F} \to \mathcal{F}^*(L)$. In the general case, the rank of λ may not be even, and thus λ may not be locally alternating. But this is the only obstruction: when the rank of λ is even we will show that λ is locally alternating, and we will show how to reduce to the even rank case.

We may assume that X is connected. Then λ is either everywhere of even rank or everywhere of odd rank because the kernel of $\lambda(x) : \mathcal{E}(x) \to \mathcal{F}^*(L)(x)$ is $\mathcal{E}(x) \cap \mathcal{F}(x)$, which is of constant rank modulo 2 by Proposition 1.1. If λ is everywhere of odd rank, then replace the Lagrangian subbundles $\mathcal{E}$, $\mathcal{F}$ of $\mathcal{V}$ by the Lagrangian subbundles $\mathcal{E}_1 := \mathcal{E} \oplus \mathcal{O}_X$ and $\mathcal{F}_1 := \mathcal{F} \oplus L$ of the orthogonal bundle $\mathcal{V}_1 := \mathcal{V} \oplus \mathcal{O}_X \oplus L$. This replaces λ by

$$\lambda_1 : \mathcal{E}_1 = \mathcal{E} \oplus \mathcal{O}_X \xrightarrow{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)} \mathcal{F}^*(L) \oplus \mathcal{O}_X = \mathcal{F}_1^*(L), \tag{7}$$

The rank of λ_1 is everywhere even, but its kernel and cokernel are the same as those of λ. Notice also that $\mathcal{E}_1(x) \cap \mathcal{F}_1(x) = \mathcal{E}(x) \cap \mathcal{F}(x)$ for all $x \in X$. Thus by replacing λ by λ_1 if necessary, we can reduce to the case where the rank is everywhere even.

Proposition 2.3. *The following are equivalent*:

(a) *The rank of λ is even everywhere.*

(b) $\dim_{k(x)} \big[\mathcal{E}(x) \cap \mathcal{F}(x)\big] \equiv \mathrm{rk}(\mathcal{E}) \pmod 2$ *for all $x \in X$.*

(c) *λ is locally alternating, i.e. there exists a cover of X by open subsets U and isomorphisms $\iota_U : \mathcal{F}|_U \cong \mathcal{E}|_U$ such that the compositions $\lambda|_U \circ \iota_U$ are alternating.*

Proof. The equivalence of (a) and (b) follows from the fact that of $\ker \lambda(x) = \mathcal{E}(x) \cap \mathcal{F}(x)$. The implication (c) $\Rightarrow$ (a) is standard. To prove (a) $\Rightarrow$ (c), use Proposition 2.1 to cover X by open subsets U over each of which $\mathcal{E}|_U, \mathcal{F}|_U$ have a common Lagrangian complement $\mathcal{M}_U$. Then by Corollary 1.3 there exist alternating maps $\zeta_U : \mathcal{F}|_U \to \mathcal{F}^*(L)|_U$ such that $\phi_{\mathcal{F}|_U, \mathcal{M}_U}(\mathcal{E}|_U) = \Gamma_{\zeta_U}$. Let $\alpha : \mathcal{E}|_U \cong \Gamma_{\zeta_U}$ be the isomorphism induced by $\phi_{\mathcal{F}|_U, \mathcal{M}_U}$, and let $\pi_1 :$

$\Gamma_{\zeta_U} \cong \mathcal{F}|_U$ and $\pi_2 : \Gamma_{\zeta_U} \to \mathcal{F}^*(L)|_U$ be the two projections from $\Gamma_{\zeta_U} \subset (\mathcal{F} \oplus \mathcal{F}^*(L))|_U$.

$$\begin{tikzcd} \mathcal{E}|_U \arrow[r, "\cong", "\alpha"'] \arrow[dr, "\lambda|_U"'] & \Gamma_{\zeta_U} \arrow[r, "\cong", "\pi_1"'] \arrow[d, "\pi_2"] & \mathcal{F}|_U \arrow[dl, "\zeta_U"] \\ & \mathcal{F}^*(L)|_U & \end{tikzcd} \tag{8}$$

Then $\iota_U := (\pi_1 \circ \alpha)^{-1}$ is an isomorphism such that $\zeta_U = \lambda|_U \circ \iota_U$ is alternating. □

Independence of the common Lagrangian complement. Unfortunately, the construction which makes λ locally alternating depends on choices of local common Lagrangian complements. We now look at what happens if we replace one choice by another.

Lemma 2.4. *Let $\mathcal{E}, \mathcal{F} \subset (\mathcal{V}, q)$ be Lagrangian subbundles, and let $\mathcal{M}$ and $\mathcal{N}$ both be common Lagrangian complements to $\mathcal{E}$ and $\mathcal{F}$. Suppose that the map $\phi_{\mathcal{F},\mathcal{M}}$ of* (5) *sends $\mathcal{E}$ to the graph of $\zeta : \mathcal{F} \to \mathcal{F}^*(L)$, and sends $\mathcal{N}$ to the graph of $h : \mathcal{F}^*(L) \to \mathcal{F}$. Then*
(a) *The maps ζ and h are alternating.*
(b) *The map $u := 1 - h\zeta$ and its transpose $u^* = 1 - \zeta h$ are both invertible.*
(c) *The isometry $\phi_{\mathcal{F},\mathcal{N}}$ sends $\mathcal{E}$ to the graph of the morphism*

$$\zeta u^{-1} = \left(u^{-1}\right)^* (\zeta - \zeta h \zeta) u^{-1}.$$

Proof. Part (a) follows from Corollary 1.3.

(b) Since $\mathcal{N}$ and $\mathcal{E}$ are complementary, their images $\Gamma_\zeta, \Gamma_h \subset \mathcal{F} \oplus \mathcal{F}^*(L)$ are also complementary. Thus the map

$$\mathcal{F} \oplus \mathcal{F}^*(L) \xrightarrow{\left(\begin{smallmatrix} 1 & h \\ \zeta & 1 \end{smallmatrix}\right)} \mathcal{F} \oplus \mathcal{F}^*(L)$$

is an isomorphism. This is equivalent to the composition

$$\begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & h \\ \zeta & 1 \end{pmatrix} = \begin{pmatrix} 1 - h\zeta & 0 \\ \zeta & 1 \end{pmatrix}$$

being invertible, or to $u = 1 - h\zeta$ being invertible. It follows that $u^* = 1 - \zeta h$ is also invertible.

(c) We now have isometries

$$\mathcal{F} \oplus \mathcal{F}^*(L) \xleftarrow[\cong]{\phi_{\mathcal{F},\mathcal{M}}} \mathcal{V} \xrightarrow[\cong]{\phi_{\mathcal{F},\mathcal{N}}} \mathcal{F} \oplus \mathcal{F}^*(L). \tag{9}$$

The left-to-right composition is the identity on the first summand $\mathcal{F}$, and sends Γ_h (corresponding to $\mathcal{N}$ on the left) onto the second summand $\mathcal{F}^*(L)$ (corresponding to $\mathcal{N}$ on the right), compatibly with the hyperbolic quadratic form on $\mathcal{F} \oplus \mathcal{F}^*(L)$. Consequently, the left-to-right composition is $\left(\begin{smallmatrix} 1 & -h \\ 0 & 1 \end{smallmatrix}\right)$. To find the image of $\mathcal{E}$ on the right, one first goes to the left (where its image

is Γ_ζ), and then applies the left-to-right composition. Therefore the image of $\mathcal{E}$ on the right is the composite image

$$\begin{tikzcd}
\mathcal{F} \arrow[r, hook, "\binom{1}{\zeta}"] & \mathcal{F} \oplus \mathcal{F}^*(L) \arrow[r, "\left(\begin{smallmatrix} 1 & -h \\ 0 & 1 \end{smallmatrix}\right)", "\text{left-to-right}"'] & \mathcal{F} \oplus \mathcal{F}^*(L). \\
\mathcal{E} \arrow[u, "\cong"] \arrow[ur, hook, "\text{left}|_{\mathcal{E}}"'] & &
\end{tikzcd}$$

The above composite map is $\left(\begin{smallmatrix} 1 & -h \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ \zeta \end{smallmatrix}\right) = \left(\begin{smallmatrix} u \\ \zeta \end{smallmatrix}\right) = \begin{pmatrix} 1 \\ \zeta u^{-1} \end{pmatrix} u$, so the image of $\mathcal{E}$ on the right is $\Gamma_{\zeta u^{-1}}$. But

$$\zeta u^{-1} = \left(u^{-1}\right)^* \left(u^* \zeta\right) u^{-1} = \left(u^{-1}\right)^* \left(\zeta - \zeta h \zeta\right) u^{-1},$$

and this completes the proof. □

Degeneracy Loci. Let $\mathcal{F}$ be a vector bundle of constant rank on a scheme X, let L be a line bundle, and let $\zeta : \mathcal{F} \to \mathcal{F}^*(L)$ be an alternating map. If $k \geq 0$ is an integer, and if $m := \mathrm{rk}(\mathcal{F}) - 2k$, then the degeneracy locus

$$Z_m(\zeta) := \{x \in X \mid \mathrm{rk}(\zeta(x)) \leq \mathrm{rk}(\mathcal{F}) - m = 2k\} \tag{10}$$

has codimension at most $m(m-1)/2$, its "expected" value. The natural scheme structure on $Z_m(\zeta)$ is defined locally by the ideal $\mathrm{Pf}_{2k+2}(\zeta)$ generated by the $(2k+2)\times(2k+2)$ Pfaffians of the alternating map ζ. These loci have been studied notably in Harris-Tu [20], and from a different point of view in Okonek [28] and Walter [34].

We need to extend this notion to the case of a locally alternating map, as described in Proposition 2.3. Let $\mathcal{E}, \mathcal{F} \subset (\mathcal{V}, q)$ be Lagrangian subbundles of a twisted orthogonal bundle, let m be an integer such that $m \equiv \dim_{k(x)} [\mathcal{E}(x) \cap \mathcal{F}(x)] \pmod 2$ for all $x \in X$, and let

$$Z_m(\mathcal{E}, \mathcal{F}) := \{x \in X \mid \dim_{k(x)}[\mathcal{E}(x) \cap \mathcal{F}(x)] \geq m\}. \tag{11}$$

The fundamental classes of these loci are discussed in Fulton-Pragacz [17] Chap. 6, where they are given as polynomials in the Chern classes of $\mathcal{E}$, $\mathcal{F}$, and L.

We now define a scheme structure on these Lagrangian degeneracy loci. Replacing λ by the λ_1 of (7) if necessary, we may assume that λ is everywhere of even rank. By Proposition 2.3 λ is then locally alternating, i.e. there exists a cover of X by open subsets U and isomorphisms $\iota_U : \mathcal{F}|_U \cong \mathcal{E}|_U$ such that the compositions $\zeta_U = \lambda|_U \circ \iota_U$ are alternating. The scheme $Z_m(\mathcal{E}, \mathcal{F})|_U$ is then defined by

$$\mathcal{I}_{Z_m(\mathcal{E},\mathcal{F})}|_U := \mathrm{Pf}_{\mathrm{rk}(\mathcal{E})-m+2}(\zeta_U). \tag{12}$$

Since $Z_m(\mathcal{E}, \mathcal{F})|_U$ is the degeneracy locus of an alternating map, its codimension is at most $m(m-1)/2$. Now the construction of the maps ζ_U in Proposition 2.3 depended on the choice of a common Lagrangian complement to $\mathcal{E}|_U$ and $\mathcal{F}|_U$. Nevertheless, the local degeneracy loci $Z_m(\mathcal{E}, \mathcal{F})|_U$

are independent of this choice and therefore glue together to form a scheme $Z_m(\mathcal{E}, \mathcal{F})$ because of Lemma 2.4 and the next lemma.

Lemma 2.5. *Let $\mathcal{F}$ be a vector bundle of constant rank and L a line bundle over a scheme X. If $\zeta : \mathcal{F} \to \mathcal{F}^*(L)$ and $h : \mathcal{F}^*(L) \to \mathcal{F}$ are alternating maps such that $u := 1 - h\zeta$ is invertible, then $\mathrm{Pf}_{2k}(\zeta) = \mathrm{Pf}_{2k}(\zeta - \zeta h \zeta)$ for all integers k.*

Proof. It is enough to prove the lemma in the case where $L = \mathcal{O}_X$ and where X is universal. So let $r := \mathrm{rk}(\mathcal{F})$, and let $R := \mathbb{Z}[X_{ij}, Y_{ij}]$ be the polynomial ring in the independent variables X_{ij}, Y_{ij} $(1 \le i < j \le r)$. Let ζ and h be the $r \times r$ matrices with coefficients in R given by

$$\zeta_{ij} := \begin{cases} X_{ij} & \text{if } i < j, \\ 0 & \text{if } i = j, \\ -X_{ji} & \text{if } i > j, \end{cases} \qquad h_{ij} := \begin{cases} Y_{ij} & \text{if } i < j, \\ 0 & \text{if } i = j, \\ -Y_{ji} & \text{if } i > j, \end{cases}$$

let $u := 1 - h\zeta$, let $\delta := \det(u)$, and set $R_\delta := R[\frac{1}{\delta}]$. Thus $X := \mathrm{Spec}(R_\delta)$. We have to show that the two ideals

$$I := \mathrm{Pf}_{2k}(\zeta), \qquad J := \mathrm{Pf}_{2k}(\zeta - \zeta h \zeta)$$

in R_δ coincide. However, we may notice the following three facts:

The ideal $I \subset R_\delta$ is prime. This is because the ideal of $\mathbb{Z}[X_{ij}]$ generated by the $2k \times 2k$ Pfaffians of ζ is prime (cf. Abeasis-Del Fra [1] §3), so its extensions to R (which is a polynomial algebra over over $\mathbb{Z}[X_{ij}]$) and to R_δ are also prime.

There is an involution of R_δ exchanging I and J. Since R is a polynomial algebra over $\mathbb{Z}$ in variables which are the entries of ζ and h, one can specify a morphism $f : R \to R_\delta$ by specifying alternating matrices $f(\zeta)$ and $f(h)$. Thus we may define f by

$$f(\zeta) := \zeta - \zeta h \zeta = \zeta u = u^* \zeta, \qquad f(h) := -u^{-1} h \, (u^*)^{-1}.$$

One computes that $f(u) = u^{-1}$, so $f(\delta)$ is the invertible element $1/\delta \in R_\delta$. Hence f extends uniquely to a morphism $f : R_\delta \to R_\delta$. One checks that $f(f(\zeta)) = \zeta$ and that $f(f(h)) = h$, so f is an involution. Since f exchanges ζ and $\zeta - \zeta h \zeta$, it exchanges I and J.

The ideals I and J define the same algebraic subset of $\mathrm{Spec}(R_\delta)$. This is equivalent to showing that a morphism $g : R_\delta \to K$ with K a field factors through R_δ / I if and only if it factors through R_δ / J. But giving such a g is equivalent to giving alternating matrices $g(\zeta)$ and $g(h)$ with coefficients in K such that $g(u) = 1 - g(h) g(\zeta)$ is invertible. Such a g factors through R_δ / I if and only if $\mathrm{rk}[g(\zeta)] < 2k$, and it factors through R_δ / J if and only if

$$\mathrm{rk}[g(\zeta - \zeta h \zeta)] = \mathrm{rk}[g(\zeta) g(u)] < 2k.$$

Since $g(u)$ is invertible, the two conditions are equivalent.

These three facts show that (in the generic case) I and J are prime ideals defining the same algebraic subsets. This proves that I are J are equal in the generic case, and therefore equal in all cases. This proves the lemma. $\square$

The definition of $Z_m(\mathcal{E},\mathcal{F})$ generalizes the definition of $Z_m(\zeta)$:

Lemma 2.6. *Let $\zeta : \mathcal{F} \to \mathcal{F}^*(L)$ be an alternating map of vector bundles, let $\Gamma_\zeta(\mathcal{F}) \subset \mathcal{F}\oplus\mathcal{F}^*(L)$ be its graph. For any m such that $m \equiv \operatorname{rk}(\mathcal{F}) \pmod 2$ the degeneracy loci $Z_m(\zeta)$ and $Z_m(\mathcal{F},\Gamma_\zeta(\mathcal{F}))$ are identical schemes.*

Finally, we may verify that our scheme-theoretic degeneracy loci do not change if we invert the order of our pair of Lagrangian subbundles, i.e.

$$Z_m(\mathcal{E},\mathcal{F}) = Z_m(\mathcal{F},\mathcal{E})$$

for all $m \equiv \operatorname{rk}(\mathcal{E}) \pmod 2$. Essentially, if $\mathcal{M}_U$ is a common Lagrangian complement of $\mathcal{E}|_U$ and of $\mathcal{F}|_U$ which leads to a ζ_U as in (12) such that $Z_m(\mathcal{E},\mathcal{F})|_U = Z_m(\zeta_U)$, then a computation similar to Lemma 2.4(c) leads to a natural identification $Z_m(\mathcal{F},\mathcal{E})|_U = Z_m(-\zeta_U)$. We leave the details to the reader.

Our interest in this paper is in the locus $Z := Z_3(\mathcal{E},\mathcal{F})$ in the case when it has codimension 3, the largest possible (and expected) value. The fundamental classes computed in [15], [16] and [17] agree with the scheme structures introduced here, which coincide with the scheme structures defined in [10].

All the results of this section have analogues for pairs of Lagrangian subbundles of twisted symplectic bundles. Degeneracy loci for such pairs are the degeneracy loci of locally symmetric maps. The symplectic case is slightly simpler than the orthogonal case because one does not need to worry about the parity of m or of $\dim_{k(x)}[\mathcal{E}(x)\cap\mathcal{F}(x)]$ (there is no symplectic analogue of Proposition 1.1). The details are left to the reader.

3. Lagrangian degeneracy loci are strongly subcanonical

We now prove the implication (b) $\Rightarrow$ (a) of our main Theorem 0.2.

Theorem 3.1. *Suppose that $(\mathcal{V},q)$ is a twisted orthogonal bundle over a locally Noetherian scheme X with values in a line bundle L. Suppose that $\mathcal{E},\mathcal{F} \subset (\mathcal{V},q)$ are Lagrangian subbundles such that $\dim_{k(x)}\big[\mathcal{E}(x)\cap\mathcal{F}(x)\big]$ is odd for all $x \in X$. Write $L_{\mathcal{E},\mathcal{F},\mathcal{V}} := \det(\mathcal{E})\otimes\det(\mathcal{F})\otimes\det(\mathcal{V})^{-1}$. Suppose that the submaximal minors of the composite map $\lambda : \mathcal{E} \hookrightarrow \mathcal{V} \cong \mathcal{V}^*(L) \twoheadrightarrow \mathcal{F}^*(L)$ generate an ideal sheaf $\mathcal{I}$ of grade 3 (the expected value).*

Then the ideal sheaf of the closed subscheme (cf. (12))

$$Z = Z_3(\mathcal{E},\mathcal{F}) = \{x \in X \mid \dim_{k(x)}[\mathcal{E}(x)\cap\mathcal{F}(x)] \geq 3\},$$

has grade 3 and satisfies $\mathcal{I}_Z^2 = \mathcal{I}$. The sheaf $\mathcal{O}_Z$ has locally free resolutions

$$0 \to L_{\mathcal{E},\mathcal{F},\mathcal{V}} \to \mathcal{E}(M) \xrightarrow{\lambda} \mathcal{F}^*(L\otimes M) \to \mathcal{O}_X \to \mathcal{O}_Z \to 0, \tag{13a}$$

$$0 \to L_{\mathcal{E},\mathcal{F},\mathcal{V}} \to \mathcal{F}(M) \xrightarrow{-\lambda^*} \mathcal{E}^*(L\otimes M) \to \mathcal{O}_X \to \mathcal{O}_Z \to 0, \tag{13b}$$

with M a line bundle such that $M^{\otimes 2} \cong L_{\mathcal{E},\mathcal{F},\mathcal{V}}\otimes L^{-1}$. Moreover, the natural isomorphism between (13b) and the dual of (13a) defines an isomorphism

$$\eta : \mathcal{O}_Z \xrightarrow{\cong} \mathcal{E}xt^3_{\mathcal{O}_X}(\mathcal{O}_Z, L_{\mathcal{E},\mathcal{F},\mathcal{V}}) =: \omega_{Z/X}(L_{\mathcal{E},\mathcal{F},\mathcal{V}}),$$

with respect to which Z is strongly subcanonical of codimension 3 in X (cf. Definition 0.1).

Corollary 3.2. *If, in the situation of the theorem, X is locally Gorenstein, then so is Z, and $\omega_Z \cong \omega_X(L^{-1}_{\mathcal{E},\mathcal{F},\mathcal{V}})|_Z$.*

The statement of the theorem remains true even if X is not Noetherian, provided one defines grade as in Eagon-Northcott [11] and Northcott [27]. The only difference in the proofs is that one uses the non-Noetherian generalizations of the Buchsbaum-Eisenbud structure theorems found in these references.

Proof of Theorem 3.1. Let $f : \mathcal{E} \to \mathcal{V}$ and $g : \mathcal{F} \to \mathcal{V}$ be the inclusions, and let $N = \mathcal{E} \cap \mathcal{F}$ be the kernel in the natural sequence

$$0 \to N \xrightarrow{\binom{i}{j}} \mathcal{E} \oplus \mathcal{F} \xrightarrow{(\,f\ -g\,)} \mathcal{V}.$$

If $\beta : \mathcal{V} \xrightarrow{\sim} \mathcal{V}^*(L)$ is the isomorphism induced by the quadratic form q, and $\lambda := g^*\beta f$, then we get a commutative diagram

$$\begin{tikzcd}
 & \mathcal{E} \arrow[rr, "\lambda"] \arrow[dr, hook, "f"] & & \mathcal{F}^*(L) \arrow[dr, "j^*"] & \\
N \arrow[ur, hook, "i"] \arrow[dr, hook, "j"'] & & \mathcal{V} \arrow[ur, "g^*\beta"] \arrow[dr, "f^*\beta"] & & N^{-1} \otimes L. \\
 & \mathcal{F} \arrow[rr, "\lambda^*"'] \arrow[ur, hook, "g"] & & \mathcal{E}^*(L) \arrow[ur, "i^*"'] &
\end{tikzcd} \tag{14}$$

Since the diagonals are short exact sequences, the kernels of λ and of λ^* are both equal to N. In addition, $fi = gj$.

We claim that N is a line bundle, and that the complexes

$$0 \to N \xrightarrow{i} \mathcal{E} \xrightarrow{\lambda} \mathcal{F}^*(L) \xrightarrow{j^*} N^{-1} \otimes L \tag{15a}$$

$$0 \to N \xrightarrow{j} \mathcal{F} \xrightarrow{-\lambda^*} \mathcal{E}^*(L) \xrightarrow{i^*} N^{-1} \otimes L \tag{15b}$$

are exact and are locally free resolutions of $\mathcal{O}_Z(N^{-1} \otimes L)$ for the subscheme $Z = Z_3(\mathcal{E}, \mathcal{F}) \subset X$ of grade 3, with $\mathcal{I}_Z^2 = \mathcal{I}$. We will prove these claims locally by making λ locally alternating and applying the Buchsbaum-Eisenbud structure theorem [6].

Now the vector bundles $\mathcal{E}$ and $\mathcal{F}$ may be of even or odd rank. If the rank is even, we use the same trick as in (7) and replace λ by

$$\mathcal{E} \oplus \mathcal{O}_X \xrightarrow{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)} \mathcal{F}^*(L) \oplus \mathcal{O}_X$$

without changing the kernel and cokernel of λ. Thus we may assume that $\mathcal{E}$ and $\mathcal{F}$ are of odd rank.

By hypothesis $\dim_{k(x)}\big[\mathcal{E}(x) \cap \mathcal{F}(x)\big]$ is also odd for all $x \in X$. Therefore, by Proposition 2.3, λ is locally alternating, i.e. X is covered by open subsets U over which there are isomorphisms $\iota_U : \mathcal{F}|_U \cong \mathcal{E}|_U$ such that $\zeta_U = \lambda|_U \circ \iota_U$

is alternating. Thus we see that our complexes (15a) and (15b) are locally isomorphic to complexes

$$0 \to N|_U \xrightarrow{j|_U} \mathcal{F} \xrightarrow{\zeta_U} \mathcal{F}^*(L) \xrightarrow{j^*|_U} (N^{-1} \otimes L)|_U \tag{16}$$

such that ζ_U is alternating with kernel $j|_U = \iota_U^{-1} \circ i|_U$ in the notation of the diagram (14). Now $\mathcal{F}$ is of odd rank, ζ_U is alternating, and the ideal $\mathcal{I}$ generated by its submaximal minors is of grade 3. So the Buchsbaum-Eisenbud structure theorem [6] applies. Therefore the kernel $N|_U$ is a line bundle, the map $j|_U$ is given by the submaximal Pfaffians of ζ_U, and the complex (16) is exact and is a resolution of $\mathcal{O}_Z(N^{-1} \otimes L)|_U$. We can also identify the ideal sheaf $\mathcal{I}$ generated by the submaximal minors of λ with $\mathcal{I}_Z^2$. This works because on U the sheaf $\mathcal{I}_Z|_U$ is generated by the submaximal Pfaffians $p_1, \ldots, p_n$ of the alternating map ζ_U, while $\mathcal{I}|_U$ is generated by the submaximal minors. Since the (i,j)-th submaximal minor is $\pm p_i p_j$ ([6] Appendix), we do indeed get $\mathcal{I}_Z^2|_U = \mathcal{I}|_U$. This verifies our claims.

Now we set $M := N \otimes L^{-1}$ and twist. We get two dual resolutions

$$0 \to M^{\otimes 2} \otimes L \to \mathcal{E}(M) \xrightarrow{\lambda} \mathcal{F}^*(L \otimes M) \to \mathcal{O}_X \to \mathcal{O}_Z \to 0, \tag{17a}$$

$$0 \to M^{\otimes 2} \otimes L \to \mathcal{F}(M) \xrightarrow{-\lambda^*} \mathcal{E}^*(L \otimes M) \to \mathcal{O}_X \to \mathcal{O}_Z \to 0. \tag{17b}$$

The alternating product of the determinant line bundles in each resolution is trivial, and therefore $M^{\otimes 2} \otimes L \cong L_{\mathcal{E},\mathcal{F},\mathcal{V}}$.

Let us now verify that $Z \subset X$ satisfies conditions A–D of Definition 0.1. The duality between the two resolutions of $\mathcal{O}_Z$ shows that $Z \subset X$ is relatively Cohen-Macaulay of codimension 3. The duality also induces an isomorphism $\eta : \mathcal{O}_Z \xrightarrow{\sim} \mathcal{E}xt^3_{\mathcal{O}_X}(\mathcal{O}_Z, L_{\mathcal{E},\mathcal{F},\mathcal{V}})$, making $Z \subset X$ subcanonical. Clearly $\mathcal{O}_Z$ is of finite local projective dimension. Moreover, η is the Yoneda extension class of (17a) and is thus the image of the class in $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{I}_Z, M^{\otimes 2} \otimes L)$ of

$$0 \to M^{\otimes 2} \otimes L \to \mathcal{E}(M) \to \mathcal{F}^*(L \otimes M) \to \mathcal{I}_Z \to 0.$$

So $Z \subset X$ is strongly subcanonical with respect to η. □

4. Degeneracy loci for split Lagrangian subbundles

In this section we discuss Lagrangian degeneracy loci in the "split" case where the bundles are $\mathcal{E}, \mathcal{F} \subset (\mathcal{F} \oplus \mathcal{F}^*(L), q_h)$. We prove the implications (c) $\Leftrightarrow$ (d) $\Rightarrow$ (a) of our main Theorem 0.2. We discuss the relation between this case and Pfaffian subschemes, and we show how the general case can be transformed into the split case. In [13] we use this split case to construct non-Pfaffian subcanonical subschemes of codimension 3 in $\mathbb{P}^n$.

Theorem 4.1. *Let $\mathcal{F}$ be a vector bundle of rank n and L a line bundle on a locally Noetherian scheme X. Let $\mathcal{F} \oplus \mathcal{F}^*(L)$ be the hyperbolic twisted orthogonal bundle.*

(a) *Suppose that*

$$\mathcal{E} \xhookrightarrow{\binom{\psi}{\phi}} \mathcal{F} \oplus \mathcal{F}^*(L) \tag{18}$$

is a Lagrangian subbundle such that $\dim_{k(x)} [\mathcal{E}(x) \cap \mathcal{F}^*(L)(x)]$ *is odd for all* $x \in X$. *Let* $L_{\mathcal{E},\mathcal{F}} := \det(\mathcal{E}) \otimes \det(\mathcal{F})^{-1}$.

If the sheaf of ideals $\mathcal{I}$ *generated by the submaximal minors of* ψ *is of grade* 3 (*the expected value*), *then the ideal sheaf of the closed subscheme*

$$Z = Z_3(\mathcal{E}, \mathcal{F}^*(L)) = \{x \in X \mid \dim_{k(x)} [\mathcal{E}(x) \cap \mathcal{F}^*(L)(x)] \geq 3\},$$

has grade 3 *and satisfies* $\mathcal{I}_Z^2 = \mathcal{I}$. *There is a commutative diagram with exact rows*

$$\begin{tikzcd}
0 \arrow[r] & L_{\mathcal{E},\mathcal{F}} \arrow[r] \arrow[d, equal] & \mathcal{E}(M) \arrow[r, "\psi"] \arrow[d, "\phi"'] & \mathcal{F}(M) \arrow[r] \arrow[d, "\phi^*"] & \mathcal{O}_X \arrow[r, two heads] \arrow[d, equal] & \mathcal{O}_Z \arrow[d, "\cong"', "\eta"] \\
0 \arrow[r] & L_{\mathcal{E},\mathcal{F}} \arrow[r] & \mathcal{F}^*(L \otimes M) \arrow[r, "-\psi^*"] & \mathcal{E}^*(L \otimes M) \arrow[r] & \mathcal{O}_X \arrow[r, two heads] & \mathcal{E}xt^3_{\mathcal{O}_X}(\mathcal{O}_Z, L_{\mathcal{E},\mathcal{F}})
\end{tikzcd} \tag{19}$$

with M *a line bundle on* X *such that* $M^{\otimes 2} \cong L^{-1} \otimes L_{\mathcal{E},\mathcal{F}}$ *and with* $\phi^*\psi$ *alternating. Moreover,* $\omega_{Z/X} \cong L^{-1}_{\mathcal{E},\mathcal{F}}|_Z$, *and* Z *is strongly subcanonical of codimension* 3 *in* X *with respect to* η.

(b) *Conversely, given a subscheme* Z *with locally free resolutions as in* (19) *and with* $\phi^*\psi$ *alternating, then* $\mathcal{E} \subset \mathcal{F} \oplus \mathcal{F}^*(L)$ *is a Lagrangian subbundle, and thus* $Z \subset X$ *is strongly subcanonical of codimension* 3.

Corollary 4.2. *If, in the situation of the theorem,* X *is locally Gorenstein, then so is* Z, *and* $\omega_Z \cong \omega_X(L^{-1}_{\mathcal{E},\mathcal{F}})|_Z$.

Proof of Theorem 4.1. The only things we need to show for part (a) which do not already follow from Theorem 3.1 are that (19) commutes and that $\phi^*\psi$ is alternating. But $\mathcal{E}$ is a totally isotropic subbundle, so any local section $e \in \Gamma(U, \mathcal{E})$ satisfies

$$0 = q_h(\psi(e) \oplus \phi(e)) = \langle \psi(e), \phi(e) \rangle = \langle \phi^*\psi(e), e \rangle. \tag{20}$$

Thus $\phi^*\psi$ is alternating, and the central part of diagram (19) commutes. The rest is easy and left to the reader. For Part (b) the fact that (19) is a quasi-isomorphism implies that $\mathcal{E}$ is a subbundle of $\mathcal{F} \oplus \mathcal{F}^*(L)$. It is totally isotropic by the same calculation (20). $\square$

Pfaffian subschemes. Okonek's Pfaffian subschemes [28] are the special case of the construction of Theorem 4.1 with $\mathcal{E} = \mathcal{F}^*(L)$ and $\phi = 1$. For if

$$\mathcal{E} \xhookrightarrow{\binom{\psi}{1}} \mathcal{E}^*(L) \oplus \mathcal{E}$$

is a Lagrangian subbundle, then ψ is alternating by Lemma 1.2 or by (20). So in this case the two resolutions of (19) reduce to

$$0 \to L \otimes M^{\otimes 2} \to \mathcal{E}(M) \xrightarrow{\psi} \mathcal{E}^*(L \otimes M) \to \mathcal{O}_X \to \mathcal{O}_Z \to 0,$$

with ψ alternating. Thus $Z \subset X$ is one of Okonek's Pfaffian subschemes.

From non-split to split bundles. When the orthogonal bundle $(\mathcal{V}, q)$ of Theorem 3.1 does not split as $\mathcal{F} \oplus \mathcal{F}^*(L)$, we cannot fill in the diagram (19) with direct arrows

$$\begin{tikzcd}
0 \arrow[r] & L_{\mathcal{E},\mathcal{F},\mathcal{V}} \arrow[r, "i"] \arrow[d, equal] & \mathcal{E}(M) \arrow[r, "\lambda"] \arrow[d, dotted] & \mathcal{F}^*(L \otimes M) \arrow[r, "j^*"] \arrow[d, dotted] & \mathcal{O}_X \arrow[r, two heads] \arrow[d, equal] & \mathcal{O}_Z \arrow[d, "\eta", "\cong"'] \\
0 \arrow[r] & L_{\mathcal{E},\mathcal{F},\mathcal{V}} \arrow[r, "j"] & \mathcal{F}(M) \arrow[r, "-\lambda^*"] & \mathcal{E}^*(L \otimes M) \arrow[r, "i^*"] & \mathcal{O}_X \arrow[r, two heads] & \mathcal{E}xt^3_{\mathcal{O}_X}(\mathcal{O}_Z, L_{\mathcal{E},\mathcal{G}}).
\end{tikzcd} \tag{21}$$

Nevertheless, by modifying the orthogonal bundle and its Lagrangian subbundles, we can usually realize the same degeneracy locus as a Lagrangian degeneracy locus of a split orthogonal bundle.

We know two strategies to accomplish this under different hypotheses. One is to apply the converse structure theorem (Theorem 6.1). The other strategy works whenever the quadratic form $q \in \Gamma(X, (S^2\mathcal{V}^*)(L))$ is the image of an $\alpha \in \Gamma(X, (\mathcal{V}^* \otimes \mathcal{V}^*)(L))$, for instance if $2 \in \Gamma(X, \mathcal{O}_X)^\times$. Then the orthogonal direct sum $(\mathcal{V}, q) \perp (\mathcal{V}, -q)$ is hyperbolic because of the inverse isometries

$$(\mathcal{V}, q) \perp (\mathcal{V}, -q) \underset{\left(\begin{smallmatrix} \beta^{-1}\alpha^* & \beta^{-1} \\ \beta^{-1}\alpha & -\beta^{-1} \end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix} 1 & 1 \\ \alpha & -\alpha^* \end{smallmatrix}\right)}{\rightleftarrows}} \mathcal{V} \oplus \mathcal{V}^*(L)$$

with $\beta := \alpha + \alpha^*$ the nonsingular symmetric bilinear form associated to q.

The composite map $\mathcal{E} \oplus \mathcal{F} \hookrightarrow (\mathcal{V}, q) \perp (\mathcal{V}, -q) \cong \mathcal{V} \oplus \mathcal{V}^*(L)$, or more explicitly

$$\mathcal{E} \oplus \mathcal{F} \xrightarrow{\left(\begin{smallmatrix} f & g \\ \alpha f & -\alpha^* g \end{smallmatrix}\right)} \mathcal{V} \oplus \mathcal{V}^*(L)$$

embeds $\mathcal{E} \oplus \mathcal{F}$ as a Lagrangian subbundle of the hyperbolic bundle $\mathcal{V} \oplus \mathcal{V}^*(L)$. We may then fill in the diagram (21) with a sequence of quasi-isomorphisms going in both directions:

$$\begin{tikzcd}
0 \arrow[r] & L_{\mathcal{E},\mathcal{F},\mathcal{V}} \arrow[r] & \mathcal{E}(M) \arrow[r, "g^*\beta f"] & \mathcal{F}^*(L \otimes M) \arrow[r] & \mathcal{O}_X \\
0 \arrow[r] & L_{\mathcal{E},\mathcal{F},\mathcal{V}} \arrow[r] \arrow[u, equal] \arrow[d, equal] & (\mathcal{E} \oplus \mathcal{F})(M) \arrow[r, "(f\ g)"] \arrow[u, "(1\ 0)"] \arrow[d, "(\alpha f\ -\alpha^* g)"'] & \mathcal{V}(M) \arrow[r] \arrow[u, "g^*\beta"'] \arrow[d, "\left(\begin{smallmatrix} f^*\alpha^* \\ -g^*\alpha^* \end{smallmatrix}\right)"] & \mathcal{O}_X \arrow[u, equal] \arrow[d, equal] \\
0 \arrow[r] & L_{\mathcal{E},\mathcal{F},\mathcal{V}} \arrow[r] & \mathcal{V}^*(L \otimes M) \arrow[r, "\left(\begin{smallmatrix} -f^* \\ -g^* \end{smallmatrix}\right)"'] & (\mathcal{E}^* \oplus \mathcal{F}^*)(L \otimes M) \arrow[r] & \mathcal{O}_X \\
0 \arrow[r] & L_{\mathcal{E},\mathcal{F},\mathcal{V}} \arrow[r] \arrow[u, equal] & \mathcal{F}(M) \arrow[r, "-f^*\beta g"'] \arrow[u, hook, "\beta g"] & \mathcal{E}^*(L \otimes M) \arrow[r] \arrow[u, hook, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] & \mathcal{O}_X \arrow[u, equal]
\end{tikzcd}$$

Another way of looking at this is to say: let $\mathcal{P}_\bullet$ and $\mathcal{Q}_\bullet$ denote the first two lines of the last diagram. If one has $\mathcal{V} \cong \mathcal{F} \oplus \mathcal{F}^*(L)$ as in Theorem 4.1, then the chain map of that theorem is induced by a twisted shifted nonsingular quadratic form on the chain complex $\mathcal{P}_\bullet$ given by a chain map $D_2(\mathcal{P}_\bullet) \to L_{\mathcal{E},\mathcal{F},\mathcal{V}}[3]$. In general, no such chain map exists, but if we can lift q to α as above, then there is a pair of chain maps

$$D_2(\mathcal{P}_\bullet) \xleftarrow{\sim} D_2(\mathcal{Q}_\bullet) \to L_{\mathcal{E},\mathcal{F},\mathcal{V}}[3] \tag{22}$$

with the first arrow a quasi-isomorphism. This means that in Theorem 3.1, we are also dealing with a sort of twisted shifted nonsingular quadratic form on $\mathcal{P}_\bullet$, but only in the derived category.

5. Local equations for the degeneracy locus

Let $Z \subset X$ be a subcanonical subscheme of codimension 3 which is a split Lagrangian degeneracy locus as in diagram (19). We give two strategies for computing equations which define this degeneracy locus locally. The first is based on the idea of using a common Lagrangian complement to make λ alternating as in Proposition 2.3. The second is based on finding standard local forms for a pair of Lagrangian submodules.

Strategy 1: Alternating homotopies. We start with a pair of Lagrangian subbundles $\mathcal{E}$ and $\mathcal{F}^*(L)$ of a twisted orthogonal bundle. We may reduce to the case where the rank of $\mathcal{E}$ and $\mathcal{F}$ are odd using (7). Also since we are working locally, we may assume that the orthogonal bundle splits as $\mathcal{F} \oplus \mathcal{F}^*(L)$ because over an affine scheme any Lagrangian subbundle has a Lagrangian complement (e.g. Knus [24] Remark I.5.5.4). Then locally $\mathcal{E}$ and $\mathcal{F}^*(L)$ have a common Lagrangian complement $\mathcal{M}$ according to Proposition 2.1. This $\mathcal{M}$ is necessarily the graph of an alternating map $h : \mathcal{F} \to \mathcal{F}^*(L)$ by Lemma 1.2. We use this h as an alternating local homotopy to transform the commutative diagram on the left below into the one on the right

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\psi"] \arrow[d, "\phi"'] & \mathcal{F} \arrow[d, "\phi^*"] \arrow[dl, dotted, "h"'] \\ \mathcal{F}^*(L) \arrow[r, "-\psi^*"'] & \mathcal{E}^*(L) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E} \arrow[r, "\psi"] \arrow[d, "\phi - h\psi"'] & \mathcal{F} \arrow[d, "\phi^* + \psi^* h"] \\ \mathcal{F}^*(L) \arrow[r, "-\psi^*"'] & \mathcal{E}^*(L) \end{tikzcd} \tag{23}$$

Then $\phi - h\psi$ is an isomorphism because it is the projection of $\mathcal{E}$ onto $\mathcal{F}^*(L)$ along their common complement $\mathcal{M}$. The dual map $\phi^* + \psi^* h$ is also an isomorphism. Thus diagram (19), with a symmetric quasi-isomorphism between the resolutions of $\mathcal{O}_Z$, can be modified locally by an alternating homotopy to get a diagram (valid locally) with a symmetric isomorphism from the resolution into its dual

$$\begin{tikzcd} 0 \arrow[r] & L_{\mathcal{E},\mathcal{F}} \arrow[r] \arrow[d, equal] & \mathcal{E}(M) \arrow[r, "\psi"] \arrow[d, "\phi - h\psi"'] & \mathcal{F}(M) \arrow[r] \arrow[d, "\phi^* + \psi^* h"] & \mathcal{O}_X \arrow[r, two heads] \arrow[d, equal] & \mathcal{O}_Z \arrow[d, "\eta", "\cong"'] \\ 0 \arrow[r] & L_{\mathcal{E},\mathcal{F}} \arrow[r] & \mathcal{F}^*(L \otimes M) \arrow[r, "-\psi^*"] & \mathcal{E}^*(L \otimes M) \arrow[r] & \mathcal{O}_X \arrow[r, two heads] & \mathcal{E}xt^3_{\mathcal{O}_X}(\mathcal{O}_Z, L_{\mathcal{E},\mathcal{F}}) \end{tikzcd}$$

Thus locally $\mathcal{O}_Z$ has a symmetric resolution

$$0 \to L_{\mathcal{E},\mathcal{F}} \to \mathcal{E}(M) \xrightarrow{\mu} \mathcal{E}^*(L \otimes M) \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

where $\mu = \phi^*\psi + \psi^* h \psi$ is alternating (and is essentially the map $\mu_{\mathcal{M}}$ of (16) and Proposition 2.1). The submaximal Pfaffians of μ give local equations for the degeneracy locus Z.

The choice of another common Lagrangian complement $\mathcal{M}$ gives a different alternating homotopy h, and vice-versa. These calculations are similar to Lemma 2.4 above.

Strategy 2: Standard local forms. Suppose that R is a commutative local ring with maximal ideal $\mathfrak{m}$ and residue field $k := R/\mathfrak{m}$. Let F be a free R-module of finite rank, and equip $F \oplus F^*$ with the hyperbolic quadratic form. Suppose that $E \subset F \oplus F^*$ is a Lagrangian submodule, i.e. a totally isotropic direct summand of rank equal to that of F. Let $\psi : E \to F$ and $\phi : E \to F^*$ be the two components of the inclusion.

Lemma 5.1. *In the above situation, there exist bases of E and F and a dual basis of F^* in which the matrices of ψ and ϕ are of the form*

$$\psi = \begin{pmatrix} \beta & 0 \\ 0 & I \end{pmatrix} \qquad\qquad \phi = \begin{pmatrix} I & 0 \\ 0 & \gamma \end{pmatrix}$$

with the blocks in the two matrices of the same size, and with β and γ alternating.

Proof. We begin by choosing bases for E and F and the dual basis of F^*, so that we can treat ψ and ϕ as matrices. Since E is a direct summand of $F \oplus F^*$, the columns of the total matrix $\left(\begin{smallmatrix} \psi \\ \phi \end{smallmatrix}\right)$ are linearly independent even modulo $\mathfrak{m}$. Moreover, by (20) above $\phi^*\psi$ is an alternating matrix because $E \subset F \oplus F^*$ is a totally isotropic submodule.

We now begin a series of row and column operations on ψ and ϕ which will put them into the required form. The column operations (resp. row operations) correspond to changes of basis of E (resp. of F and F^*) and to the action of invertible matrices P (resp. Q) on ψ and ϕ via $\psi \rightsquigarrow Q^{-1}\psi P$ and $\phi \rightsquigarrow Q^*\phi P$.

Choose a maximal invertible minor of ϕ. After row and column operations, we can assume that the corresponding submatrix is an identity block lying in the upper left corner of ϕ and that the blocks below and to the right of it are 0. Thus we can assume that

$$\phi = \begin{pmatrix} I & 0 \\ 0 & \delta \end{pmatrix} \qquad\qquad \psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}$$

where the blocks of the two matrices are of the same size, the on-diagonal blocks are square, and the coefficients of δ lie in $\mathfrak{m}$. Since

$$\phi^*\psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \delta^*\psi_{21} & \delta^*\psi_{22} \end{pmatrix}$$

is alternating, we see that all the coefficients of ψ_{12} also lie in $\mathfrak{m}$. Hence all the coefficients in the last block of columns of $\left(\begin{smallmatrix}\psi\\ \phi\end{smallmatrix}\right)$ lie in $\mathfrak{m}$ except those in ψ_{22}. Since these columns must be linearly independent modulo $\mathfrak{m}$, it follows that ψ_{22} must be invertible. Applying a new set of column operations to ϕ and ψ, we may assume that $\psi = \left(\begin{smallmatrix}\psi_{11} & \epsilon\\ \psi_{21} & I\end{smallmatrix}\right)$ and that $\phi = \left(\begin{smallmatrix}I & 0\\ 0 & \gamma\end{smallmatrix}\right)$. Moreover, $\phi^*\psi$ remains alternating, which actually means that ψ_{11} and γ are alternating, and $\epsilon = -\psi_{21}^*\gamma$. A final set of row and column operations using the matrices $Q = \left(\begin{smallmatrix}I & \psi_{21}^*\gamma\\ 0 & I\end{smallmatrix}\right)$ and $P = \left(\begin{smallmatrix}I & 0\\ -\psi_{21} & I\end{smallmatrix}\right)$ puts ψ and ϕ into the form required by the lemma. □

Corollary 5.2. *Let R be a commutative local ring with residue field k, let F be a free R-module of odd rank, and let $E \subset F \oplus F^*$ be a Lagrangian submodule such that $\dim_k[(E\otimes k)\cap(F\otimes k)]$ is odd. Let $\psi : E \to F$ and $\phi : E \to F$ be the two components of the inclusion.*

(a) *The determinant of ϕ is of the form $\det\phi = af^2$ with a invertible.*

(b) *If $\det(\phi)$ is not a zero-divisor, and if ψ degenerates along an ideal I of height and grade 3 (as expected), then this ideal is $I = (\mathrm{Pf}(\phi^*\psi) : f)$, where $\mathrm{Pf}(\phi^*\psi)$ is the ideal generated by the submaximal Pfaffians of $\phi^*\psi$, and where f is as in part (a).*

Proof. (a) We put the matrices of ϕ and of ψ in the special form of Lemma 5.1, and we set $f := \mathrm{Pf}(\gamma)$. The determinant of the matrix of ϕ is then f^2. Consequently the determinant of the matrix of ϕ with respect to any bases of E and F is of the form af^2, with a an invertible element of R coming from the determinants of the change-of-basis matrices.

(b) Using the special forms for ϕ and ψ given in Lemma 5.1, we find that I is generated by the submaximal Pfaffians $p_1, \ldots, p_{2s+1}$ of β, while the ideal $\mathrm{Pf}(\phi^*\psi)$ is generated by $fp_1, \ldots, fp_{2s+1}$. Since we suppose $\det(\phi)$ and therefore f are not zero-divisors, this gives (b). □

6. Subcanonical subschemes are Lagrangian degeneracy loci

In this section we prove the implication (a) ⇒ (d) of our main Theorem 0.2. Taken together with Theorems 3.1 and 4.1, this proves the main theorem, because the implication (c) ⇒ (b) is trivial.

Theorem 6.1. *Let A be a Noetherian ring, and $X \subset \mathbb{P}^N_A$ a locally closed subscheme. If $Z \subset X$ is a codimension 3 strongly subcanonical subscheme (cf. Definition 0.1), then there exist vector bundles $\mathcal{E}$ and $\mathcal{G}$, a line bundle L on X, and an embedding of $\mathcal{E}$ as a Lagrangian subbundle of the twisted*

hyperbolic bundle $\mathcal{G} \oplus \mathcal{G}^*(L)$ such that $Z = Z_3(\mathcal{E}, \mathcal{G}^*(L))$ and $\mathcal{O}_Z$ has symmetrically quasi-isomorphic locally free resolutions

$$
\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r, "\psi"] \arrow[d, "\phi"'] & \mathcal{G} \arrow[r] \arrow[d, "\phi^*"] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & \mathcal{O}_Z \arrow[r] \arrow[d, "\cong"', "\eta"] & 0 \\
0 \arrow[r] & L \arrow[r] & \mathcal{G}^*(L) \arrow[r, "-\psi^*"] & \mathcal{E}^*(L) \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{E}xt^3_{\mathcal{O}_X}(\mathcal{O}_Z, L) \arrow[r] & 0
\end{tikzcd}
\tag{24}
$$

with $\phi^*\psi : \mathcal{E} \to \mathcal{E}^*(L)$ an alternating map.

We will need the following two lemmas in the proof of the theorem.

Lemma 6.2. *Let A be a Noetherian ring, let $X \subset \mathbb{P}^N_A$ be a locally closed subscheme, let $\mathcal{F}, \mathcal{G}$ be coherent sheaves on X, let $\mathcal{M}$ be a vector bundle on X, and let $p > 0$.*

(a) *If $\xi \in \operatorname{Ext}^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, then there exists a vector bundle $\mathcal{E}$ on X and a surjection $f : \mathcal{E} \twoheadrightarrow \mathcal{F}$ such that the pullback class $f^*\xi \in \operatorname{Ext}^p_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G})$ vanishes.*

(b) *If $\zeta \in \operatorname{Ext}^p_{\mathcal{O}_X}(\Lambda^2\mathcal{M}, \mathcal{G})$, then there exists a surjection of vector bundles $\mathcal{P} \twoheadrightarrow \mathcal{M}$ such that the pullback of ζ to $\operatorname{Ext}^p_{\mathcal{O}_X}(\Lambda^2\mathcal{P}, \mathcal{G})$ vanishes.*

Proof. (a) Extending $\mathcal{F}$ to a coherent sheaf on the closure $\overline{X} \subset \mathbb{P}^N_A$ and then applying Serre's Theorem A, we see that there exists a surjection of the form $g : \mathcal{O}_X(-n)^r \twoheadrightarrow \mathcal{F}$. Pulling back gives us a class $g^*\xi \in H^p(X, \mathcal{G}(n))^r$.

Let R be the homogeneous coordinate ring of $\overline{X}$, and let $I \subset R$ be the homogeneous ideal of strictly positive degree elements vanishing on the closed subset $\overline{X} \setminus X$. Extend $\mathcal{G}$ to a coherent sheaf on $\overline{X}$, and let G be a finitely generated graded R-module whose associated sheaf is this extension of $\mathcal{G}$. Then $H^p_*(X, \mathcal{G})^r \cong H^{p+1}_I(G)^r$. Consequently, $g^*\xi$, as a member of a local cohomology module, is annihilated by some power I^m of I. A finite set of homogeneous generators of I^m gives surjections $\bigoplus_i R(-a_i) \twoheadrightarrow I^m$ and $\bigoplus_i \mathcal{O}_X(-a_i) \twoheadrightarrow \mathcal{O}_X$, such that the pullback of $g^*\xi$ along the induced map $\bigoplus_i \mathcal{O}_X(-a_i - n)^r \twoheadrightarrow \mathcal{O}_X(-n)^r$ vanishes.

(b) For the same reasons as in part (a), ζ is killed by some power of I. For convenience we assume that the same I^m as in part (a) kills ζ. Let $\bigoplus_i \mathcal{O}_X(-a_i) \twoheadrightarrow \mathcal{O}_X$ be the surjection used in part (a). Then the surjection $\bigoplus_i \mathcal{M}(-a_i) \twoheadrightarrow \mathcal{M}$ kills ζ because the exterior square factors as $\Lambda^2\big(\bigoplus_i \mathcal{M}(-a_i)\big) \twoheadrightarrow \bigoplus_{i \le j} (\Lambda^2\mathcal{M})(-a_i - a_j) \twoheadrightarrow \Lambda^2\mathcal{M}$. ☐

Lemma 6.3 (Serre; [29] Lemma 5.1.2)**.** *Let A be a Noetherian local ring and M a finitely generated A-module of projective dimension at most 1. Suppose that $\zeta \in \operatorname{Ext}^1_A(M, A)$ corresponds to the extension $0 \to A \to N \to M \to 0$. Then N is a free A-module if and only if ζ generates the A-module $\operatorname{Ext}^1_A(M, A)$.*

Proof of Theorem 6.1. By hypothesis, η lifts to a class in $\operatorname{Ext}^2_{\mathcal{O}_X}(\mathcal{I}_Z, L)$. By Lemma 6.2(a) there exists a vector bundle $\mathcal{M}$ and a surjection and kernel

$0 \to \mathcal{K} \to \mathcal{M} \to \mathcal{I}_Z \to 0$ such that η lifts further to a class $\zeta \in \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{K}, L)$. This defines an extension $0 \to L \to \mathcal{E} \to \mathcal{K} \to 0$. Attaching these extensions gives an acyclic complex

$$0 \to L \to \mathcal{E} \to \mathcal{M} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0. \tag{25}$$

We claim that $\mathcal{E}$ is locally free. Our reasoning is as follows. Since the local projective dimension of $\mathcal{O}_Z$ is at most 3, the local projective dimension of $\mathcal{K}$ is at most 1. By Lemma 6.3, $\mathcal{E}$ will be locally free if ζ generates the sheaf $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{K}, L)$. Moreover the sheaves $\mathcal{O}_Z$, $\mathcal{E}xt^3_{\mathcal{O}_X}(\mathcal{O}_Z, L))$, and $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{K}, L)$ are all isomorphic, and their respective global sections 1, η, and ζ correspond under these isomorphisms.

$$\begin{tikzcd}
\zeta \in \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{K}, L) \arrow[r] \arrow[d] & H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{K}, L)) \arrow[d, "\cong"] & \\
\eta \in \operatorname{Ext}^3_{\mathcal{O}_X}(\mathcal{O}_Z, L) \arrow[r, "\cong"] & H^0(\mathcal{E}xt^3_{\mathcal{O}_X}(\mathcal{O}_Z, L)) & H^0(\mathcal{O}_Z) \ni 1 \arrow[l, "\cong"']
\end{tikzcd}$$

Since 1 generates $\mathcal{O}_Z$, the section ζ generates $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{K}, L)$. Thus $\mathcal{E}$ is locally free.

The complex

$$\mathcal{A}_\bullet : \qquad 0 \to L \to \mathcal{E} \to \mathcal{M} \to \mathcal{O}_X \to 0 \tag{26}$$

is now a locally free resolution of $\mathcal{O}_Z$. As in Buchsbaum-Eisenbud [6] and Walter [34], we try to make this into a commutative associative differential graded algebra resolution of $\mathcal{O}_Z$ by constructing a map $D_2(\mathcal{A}_\bullet) \to \mathcal{A}_\bullet$ from the divided square covering the identity in degree 0:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{M}(L) \oplus D_2\mathcal{E} \arrow[r] \arrow[d, dotted] & L \oplus (\mathcal{E} \otimes \mathcal{M}) \arrow[r] \arrow[d, dotted] & \mathcal{E} \oplus \Lambda^2\mathcal{M} \arrow[r] \arrow[d, dotted] & \mathcal{M} \arrow[r] \arrow[d, equal] & \mathcal{O}_X \arrow[d, equal] \\
\cdots \arrow[r] & 0 \arrow[r] & L \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{M} \arrow[r] & \mathcal{O}_X
\end{tikzcd} \tag{27}$$

Now $\Lambda^2\mathcal{M}$ maps into the kernel $\mathcal{K}$ of $\mathcal{M} \to \mathcal{O}_X$. Hence the first problem in filling in the dotted arrows above is to carry out a lifting

$$\begin{tikzcd}
 & & & \Lambda^2\mathcal{M} \arrow[d] \arrow[dl, dotted] & \\
0 \arrow[r] & L \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{K} \arrow[r] & 0
\end{tikzcd}$$

The obstruction to carrying out the lifting is a class $\zeta \in \operatorname{Ext}^1_{\mathcal{O}_X}(\Lambda^2\mathcal{M}, L)$. There is no reason for this class to vanish. So the liftings sought in (27) need not exist. But there is a way around this.

By Lemma 6.2(b) there is a surjection from another vector bundle $\mathcal{G} \twoheadrightarrow \mathcal{M}$ such that the pullback of ζ to $\operatorname{Ext}^1_{\mathcal{O}_X}(\Lambda^2\mathcal{G}, L)$ vanishes. We now redo the construction of the complex and get commutative diagrams with exact rows

and columns

$$
\begin{tikzcd}
 & \mathcal{R} \arrow[r, equal] \arrow[d, hook] & \mathcal{R} \arrow[d, hook] & \\
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & \mathcal{F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{K}' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{K} \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{R} \arrow[r, equal] \arrow[d, hook] & \mathcal{R} \arrow[d, hook] & & \\
0 \arrow[r] & \mathcal{K}' \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{I}_Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{M} \arrow[r] & \mathcal{I}_Z \arrow[r] & 0
\end{tikzcd}
$$

This allows us to construct a new complex

$$\mathcal{B}_{\bullet}: \qquad 0 \to L \to \mathcal{F} \xrightarrow{\psi} \mathcal{G} \to \mathcal{O}_X \to 0.$$

One sees easily that $\mathcal{R}$ and therefore $\mathcal{F}$ are also vector bundles. But this time, the composite map $\Lambda^2\mathcal{G} \to \mathcal{K}' \to \mathcal{K}$, lifts to $\mathcal{E}$ since the obstruction is the class in $\mathrm{Ext}^1_{\mathcal{O}_X}(\Lambda^2\mathcal{G}, L)$ which we got to vanish using Lemma 6.2(b). Since the square marked with the $\square$ is cartesian, we get a lifting $\Lambda^2\mathcal{G} \to \mathcal{F}$. The other liftings

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathcal{G}(L) \oplus D_2\mathcal{F} \arrow[r] \arrow[d, dotted] & L \oplus (\mathcal{F} \otimes \mathcal{G}) \arrow[r] \arrow[d, dotted] & \mathcal{F} \oplus \Lambda^2\mathcal{G} \arrow[r] \arrow[d, dotted] & \mathcal{G} \arrow[r] \arrow[d, equal] & \mathcal{O}_X \arrow[d, equal] \\
\cdots \arrow[r] & 0 \arrow[r] & L \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{O}_X
\end{tikzcd}
\tag{28}
$$

now occur automatically. We therefore get a chain map $D_2\mathcal{B}_{\bullet} \to \mathcal{B}_{\bullet}$ which makes $\mathcal{B}_{\bullet}$ into a commutative associative differential graded algebra with divided powers.

We now claim that having this differential graded algebra structure gives us all the properties we want and puts us into the situation of Theorem 4.1. Indeed, as in Buchsbaum-Eisenbud [6], the multiplication gives pairings $\mathcal{B}_i \otimes \mathcal{B}_{3-i} \to \mathcal{B}_3 = L$, and therefore maps $\mathcal{B}_i \to \mathcal{B}^*_{3-i}(L)$. These maps are compatible with the differential, and as a result, the following diagram commutes:

$$
\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & \mathcal{F} \arrow[r, "\psi"] \arrow[d, "\phi"'] & \mathcal{G} \arrow[r] \arrow[d, "\phi^*"] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & \mathcal{O}_Z \arrow[r] \arrow[d, "\simeq"', "\eta"] & 0 \\
0 \arrow[r] & L \arrow[r] & \mathcal{G}^*(L) \arrow[r, "-\psi^*"'] & \mathcal{F}^*(L) \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{E}xt^3_{\mathcal{O}_X}(\mathcal{O}_Z, L) \arrow[r] & 0
\end{tikzcd}
$$

The top row is exact by construction, and the bottom row is exact because it is the dual of the top row which is a resolution of a sheaf of grade 3. Since η is an isomorphism, one sees that

$$0 \to \mathcal{F} \xrightarrow{\binom{\psi}{\phi}} \mathcal{G} \oplus \mathcal{G}^*(L) \xrightarrow{(\phi^* \ \psi^*)} \mathcal{F}^*(L) \to 0$$

is exact. Thus $\mathcal{F}$ embeds in $\mathcal{G} \oplus \mathcal{G}^*(L)$ as a subbundle which is totally isotropic for the hyperbolic symmetric bilinear form on $\mathcal{G} \oplus \mathcal{G}^*(L)$. The subbundle $\mathcal{F}$ is even totally isotropic for the hyperbolic quadratic form, since the restriction

of this form to local sections of $\mathcal{F}$ is the function $e \mapsto \langle \phi(e), \psi(e) \rangle$, and this function vanishes because the composite map from diagram (28)

$$\begin{array}{ccccc} D_2\mathcal{F} & \longrightarrow & \mathcal{F} \otimes \mathcal{G} & \longrightarrow & L \\ f \otimes f & \longmapsto & f \otimes \psi(f) & \longmapsto & \langle \phi(f), \psi(f) \rangle \end{array}$$

factors through 0 and hence vanishes identically. Thus $\mathcal{F}$ is a Lagrangian subbundle of $\mathcal{G} \oplus \mathcal{G}^*(L)$. This completes the proof. □

7. Points in $\mathbb{P}^3$

In this and the following section we discuss several classes of examples which satisfy some or all of the conditions A-D of the definition of a strongly subcanonical subscheme and thus Theorems 0.2 and 6.1 may apply. Additional geometric applications and examples can be found in our paper [13].

Okonek [28], p. 429, has shown that any reduced set of points in $\mathbb{P}^3$ is Pfaffian. By carefully analyzing the constructions of Theorem 6.1, we will describe Pfaffian resolutions of locally Gorenstein zero-dimensional subschemes in $\mathbb{P}^3$ (see Remark 7.4).

For a locally Gorenstein zero-dimensional subscheme $Z \subset \mathbb{P}^3_k$ over a field k, there are many isomorphisms $\eta : \mathcal{O}_Z \xrightarrow{\sim} \omega_Z(t)$. Which triples $(Z, \omega_{\mathbb{P}^3}(t), \eta)$ satisfy all the conditions of Definition 0.1, and which do not? In particular (and this is the only condition which causes trouble), when does the image of η in $H^3(\mathbb{P}^3, \omega_{\mathbb{P}^3}(t))$ vanish?

We will use the following notation, see for instance [12]. Let $I \subset R := k[x_0, x_1, x_2, x_3]$ be the homogeneous ideal of Z, let $A := R/I$ be its homogeneous coordinate ring, and let $\omega_A := \operatorname{Ext}^3_R(A, R(-4))$ be its canonical module. Note that $\eta \in H^0_*(\omega_Z) \supset \omega_A$. Also if M is a graded R-module, then let M' be its dual as a graded k-vector space, endowed with the natural dual R-module structure.

Proposition 7.1. *Let $Z \subset \mathbb{P}^3$ be a locally Gorenstein subscheme of dimension zero, and $\eta : \mathcal{O}_Z \xrightarrow{\sim} \omega_Z(t)$ an isomorphism. Then the triple $(Z, \omega_{\mathbb{P}^3}(t), \eta)$ is subcanonical and satisfies conditions A-C of Definition 0.1, and it satisfies condition D if and only if $\eta \in \omega_A$.*

Proof. The map $\eta : \mathcal{O}_Z \to \omega_Z(t)$ may be identified with an element of $\operatorname{Ext}^3_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_Z, \omega_{\mathbb{P}^3}(t)) \cong H^0(\mathcal{O}_Z(-t))'$. The subscheme $Z \subset \mathbb{P}^3$ satisfies condition D for η if and only if η is in the image of $\operatorname{Ext}^2_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{I}_Z, \omega_{\mathbb{P}^3}(t)) \cong H^1(\mathcal{I}_Z(-t))'$.

Local duality and Serre duality give identifications

$$\omega_A := \operatorname{Ext}^3_R(A, R(-4)) \cong H^1_{\mathfrak{m}}(A)' \cong H^1_*(\mathcal{I}_Z)'$$

and $H^0_*(\omega_Z) \cong H^0_*(\mathcal{O}_Z)'$ which are compatible with the inclusions. So η satisfies condition D of Definition 0.1 if and only if $\eta \in \omega_A$. □

Theorem 7.2. *Let $Z \subset \mathbb{P}^3$ be a locally Gorenstein subscheme of dimension 0, and let $\eta \in H^0(\omega_Z(t))$. Suppose that* (a) *$\eta$ generates the sheaf ω_Z,* (b) *$\eta \in \omega_A$, and* (c) *if $t = -2\ell$ is even, then the following nondegenerate symmetric bilinear form on $H^0(\mathcal{O}_Z(\ell))$ is metabolic* (*i.e. contains a Lagrangian subspace*):

$$H^0(\mathcal{O}_Z(\ell)) \times H^0(\mathcal{O}_Z(\ell)) \to H^0(\mathcal{O}_Z(2\ell)) \xrightarrow{\eta} H^0(\omega_Z) \xrightarrow{\mathrm{tr}} k. \tag{29}$$

Then there exists a locally free resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(t-4) \to \mathcal{F}^*(t-4) \xrightarrow{\psi} \mathcal{F} \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_Z \to 0 \tag{30}$$

with ψ alternating and $\mathcal{I}_Z$ generated by the submaximal Pfaffians of ψ and such that the Yoneda extension class of (30) *is $\eta \in \mathrm{Ext}^3_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_Z, \mathcal{O}_{\mathbb{P}^3}(t-4)) \cong H^0(\omega_Z(t))$.*

Conversely, if there exists a locally free resolution of $\mathcal{O}_Z$ as in (30) *with ψ alternating, then its Yoneda extension class η satisfies conditions* (a), (b) *and* (c) *above.*

In order for the symmetric bilinear form (29) to be metabolic, it is necessary for $\deg(Z)$ to be even. If the base field k is closed under square roots, this is also sufficient.

In any case, the conditions of the theorem always hold if t is large and odd and η is general. This proves the following result, which was proven for reduced sets of points by Okonek [28], p. 429.

Corollary 7.3. *A zero-dimensional subscheme of $\mathbb{P}^3$ is Pfaffian if and only if it is locally Gorenstein.*

Proof of Theorem 7.2. We show how to start the proof off. But we will stop when we reach the point where it becomes identical to the proof of the main result of [34].

Suppose that Z, t, η satisfy conditions (a), (b), and (c) of the theorem. Condition (a) implies that the map $\eta : \mathcal{O}_Z \to \omega_Z(t)$ is an isomorphism. So η and Serre duality induce a symmetric perfect pairing

$$H^0_*(\mathcal{O}_Z) \times H^0_*(\mathcal{O}_Z) \xrightarrow{\mathrm{mult}} H^0_*(\mathcal{O}_Z) \xrightarrow{\eta} H^0_*(\omega_Z(t)) \xrightarrow{\mathrm{tr}} k(t) \tag{31}$$

which pairs $H^0(\mathcal{O}_Z(n))$ with $H^0(\mathcal{O}_Z(-n-t))$ for all n.

Condition (c) implies that $H^0_*(\mathcal{O}_Z)$ contains a Lagrangian submodule M for this symmetric perfect pairing. Indeed if t is odd, one can pick $M := \bigoplus_{n>-t/2} H^0(\mathcal{O}_Z(n))$. If t is even, then there exists a Lagrangian subspace $W \subset H^0(\mathcal{O}_Z(-t/2))$, and one can pick $M := W \oplus \bigoplus_{n>-t/2} H^0(\mathcal{O}_Z(n))$.

The two submodules $A \subset H^0_*(\mathcal{O}_Z)$ and $\omega_A \subset H^0_*(\omega_Z)$ are orthogonal complements of each other under the Serre duality pairing; see for example [12]. Hence condition (b), that $\eta \in \omega_A$, implies that $\eta A \subset \omega_A$ and therefore that $A = \omega_A^\perp \subset (\eta A)^\perp$. Now the orthogonal complement of $\eta A \subset H^0_*(\omega_Z)$ under the Serre duality pairing corresponds to the orthogonal complement of

$A \subset H^0_*(\mathcal{O}_Z)$ under our pairing (31). So condition (b) implies that $A \subset A^\perp$. In other words $A \subset H^0_*(\mathcal{O}_Z)$ is sub-Lagrangian.

It now follows that there exists a Lagrangian submodule L such that $0 \subset A \subset L = L^\perp \subset A^\perp \subset H^0_*(\mathcal{O}_Z)$. For instance, pick $L := A + (M \cap A^\perp)$ (cf. Knus [24] Lemma I.6.1.2).

One easily checks that $A_n = (A^\perp)_n = H^0(\mathcal{O}_Z(n))$ for $n \gg 0$, and that $A_n = (A^\perp)_n = 0$ for $n \ll 0$. Consequently $A^\perp/A$ is of finite length. It has an induced nondegenerate symmetric bilinear form, and it has a Lagrangian submodule L/A.

We now claim that we can construct a locally free resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(t-4) \xrightarrow{\alpha} \mathcal{F}^*(t-4) \xrightarrow{\psi} \mathcal{F} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_Z \to 0$$

with ψ alternating and such that $H^1_*(\mathcal{F}) \cong L/A$, and $H^2_*(\mathcal{F}) = 0$. Moreover, β induces a surjection $H^0_*(\mathcal{F}) \twoheadrightarrow H^0_*(\mathcal{I}_Z)$. Different pieces of the resolution contribute different pieces of the cohomology module $H^0_*(\mathcal{O}_Z)$. The submodule A is contributed by $\operatorname{coker} H^0_*(\beta)$; the piece L/A by $H^1_*(\mathcal{F})$; the piece $A^\perp/L$ by $H^2_*(\mathcal{F}^*(t-4))$; and the piece $H^0_*(\mathcal{O}_Z)/A^\perp$ is contributed by $\ker H^3_*(\alpha)$.

The construction of this resolution and the verification of its properties can be done using the Horrocks correspondence by the same method as in Walter [34]. It is quite long and we omit the details. □

Remark 7.4. The graded module $A^\perp/A$ above can be thought of as the "intermediate cohomology" or *deficiency module* of (Z, t, η). To emphasize the dependence of this module on η, one could write it as $(\eta A)^\perp/A$, where $(\eta A)^\perp \subset H^0_*(\mathcal{O}_Z)$ means the orthogonal complement of $\eta A \subset H^0_*(\omega_Z)$ with respect to the Serre duality pairing. Now $(\eta A)^\perp/A$ is dual to $(\omega_A/\eta A)$, and it is also self-dual with a shift. Consequently if $\eta \in \omega_A$ is of degree t, then the corresponding deficiency module is

$$(\eta A)^\perp/A \cong (\omega_A/\eta A)' \cong (\omega_A/\eta A)(t).$$

In Theorem 7.2 we split the deficiency module in half, and put a Lagrangian subhalf in $\mathcal{F}$ and the quotient half in $\mathcal{F}^*(t-4)$. The Pfaffian resolutions of $\mathcal{O}_Z$ are thus classified up to symmetric homotopy equivalence by pairs $(\eta, L/A)$ with $\eta \in \omega_A$ generating the sheaf ω_Z, and with $L/A \subset (\eta A)^\perp/A$ a Lagrangian submodule.

An alternative strategy for dealing with this deficiency module is to construct a diagram of the form of (19) in Theorem 4.1 (we write $\mathcal{O} := \mathcal{O}_{\mathbb{P}^3}$ to try to stay inside the margins):

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(t-4) \arrow[r] \arrow[d, equal] & \mathcal{G} \arrow[r, "\psi"] \arrow[d, "\phi"'] & \bigoplus \mathcal{O}(-a_i) \arrow[r] \arrow[d, "\phi^*"] & \mathcal{O} \arrow[r, two heads] \arrow[d, equal] & \mathcal{O}_Z \arrow[d, "\eta", "\cong"'] \\
0 \arrow[r] & \mathcal{O}(t-4) \arrow[r] & \bigoplus \mathcal{O}(a_i+t-4) \arrow[r, "-\psi^*"'] & \mathcal{G}^*(t-4) \arrow[r] & \mathcal{O} \arrow[r, two heads] & \omega_Z(t)
\end{tikzcd} \tag{32}$$

with $\bigoplus \mathcal{O}(-a_i)$ corresponding to a minimal set of generators of the homogeneous ideal of Z, with $H^2_*(\mathcal{G}) \cong (\eta A)^\perp/A$, the deficiency module, and with $H^1_*(\mathcal{G}) = 0$.

We now give examples both of Pfaffian resolutions as in (30) which split the deficiency module, and of resolutions as in (32) in the form of Theorem 4.1 which gather the deficiency module up in one piece.

Example 1: one point. Consider a single rational point Q. Its geometry is simple, but we can make its algebra complicated.

The canonical module of Q is $\omega_A \cong \bigoplus_{n\geq 1} H^0(\omega_Q(n))$. If we pick a nonzero η of degree 1, then it generates ω_A, and its deficiency module vanishes. The constructions described above both lead unsurprisingly to the Koszul resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_Q \to 0.$$

However, if we let $\eta \in \omega_A$ be a nonzero element of degree 2, then the deficiency module is k concentrated in degree -1, and the construction (32) yields a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-2) \arrow[r] \arrow[d, equal] & \Omega^2_{\mathbb{P}^3}(1) \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 3} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^3} \arrow[r, two heads] \arrow[d, equal] & \mathcal{O}_Q \arrow[d, "\cong"', "\eta"] \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-2) \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 3} \arrow[r] & \Omega_{\mathbb{P}^3}(1) \arrow[r] & \mathcal{O}_{\mathbb{P}^3} \arrow[r, two heads] & \omega_Q(2).
\end{tikzcd}$$

More generally, if we let $\eta \in \omega_A$ be a nonzero element of degree t, then the deficiency module is $\bigoplus_{n=-(t-1)}^{-1} H^0(\mathcal{O}_Q(n))$, and the construction (32) yields

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(t-4) \arrow[r] \arrow[d, equal] & \mathcal{F}^*_t(t-4) \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 3} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^3} \arrow[r, two heads] \arrow[d, equal] & \mathcal{O}_Q \arrow[d, equal] \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(t-4) \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(t-3)^{\oplus 3} \arrow[r] & \mathcal{F}_t \arrow[r] & \mathcal{O}_{\mathbb{P}^3} \arrow[r, two heads] & \mathcal{O}_Q
\end{tikzcd}$$

with $\mathcal{F}_t$ a rank 3 locally free sheaf which is the sheafification of the kernel of the presentation of the deficiency module:

$$0 \to \mathcal{F}_t \to \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(t-2)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^3}(t-1) \to 0.$$

If one lets $\eta \in \omega_A$ be a nonzero element of degree 3, then applying the methods of Theorem 7.2 yields a resolution which one recognizes as the Koszul complex associated to the zero locus of a section of the rank 3 bundle $\mathcal{T}_{\mathbb{P}^3}(-1)$:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \Omega^2_{\mathbb{P}^3}(2) \to \Omega_{\mathbb{P}^3}(1) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_Q \to 0.$$

Example 2: three points. If Z is the union of three non-collinear rational points, then the module ω_A has two generators of degree 0, and Z is not arithmetically Gorenstein. If we pick a general $\eta \in \omega_A$ of degree 0, then the deficiency module is k, concentrated in degree 0, and the construction (32)

yields a diagram (in which we again write $\mathcal{O} := \mathcal{O}_{\mathbb{P}^3}$ in order to simplify the notation):

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-4) \arrow[r] \arrow[d, equal] & \mathcal{O}(-3) \oplus \Omega^2_{\mathbb{P}^3} \arrow[r] \arrow[d] & \mathcal{O}(-2)^{\oplus 3} \oplus \mathcal{O}(-1) \arrow[r] \arrow[d] & \mathcal{O} \arrow[r, two heads] \arrow[d, equal] & \mathcal{O}_Z \arrow[d, equal] \\
0 \arrow[r] & \mathcal{O}(-4) \arrow[r] & \mathcal{O}(-3) \oplus \mathcal{O}(-2)^{\oplus 3} \arrow[r] & \Omega_{\mathbb{P}^3} \oplus \mathcal{O}(-1) \arrow[r] & \mathcal{O} \arrow[r, two heads] & \mathcal{O}_Z
\end{tikzcd}$$

If we pick a general $\eta \in \omega_A$ of degree 1, then the deficiency module $(\omega_A/\eta A)(1)$ is of length 4, concentrated in degrees 0 and -1, and the methods of Theorem 7.2 yield a symmetric resolution (with alternating middle map ψ):

$$0 \to \mathcal{O}(-3) \to \Omega^2_{\mathbb{P}^3}(1)^{\oplus 2} \oplus \mathcal{O}(-2) \xrightarrow{\psi} \Omega^{\oplus 2}_{\mathbb{P}^3} \oplus \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_Z \to 0.$$

8. Some weakly subcanonical subschemes

In this section we give some examples of weakly subcanonical subschemes. These are examples of subschemes $Z \subset X$ which satisfy conditions A-B of the definition of a strongly subcanonical subscheme but fail one or both of conditions C-D. Thus the Serre construction (in codimension 2) and our Theorem 6.1 (in codimension 3) fail for these subschemes.

A weakly subcanonical curve. We construct a subcanonical curve $C \subset \mathbb{P}^1 \times \mathbb{P}^n$ for $n \geq 2$ which fails the lifting condition D of Definition 0.1.

Let C be a nonsingular projective curve of genus 2 over an algebraically closed field k, let P be one of its Weierstrass points, and let D be a divisor of degree 4 on C. A base-point-free pencil in the linear system of divisors $|D|$ defines a map $f : C \to \mathbb{P}^1$, and a base-point-free net in $|D + P|$ defines a map $g : C \to \mathbb{P}^2$. Composing g with a linear embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^n$ gives a map $h : C \to \mathbb{P}^n$. Let $i := (f, h) : C \to \mathbb{P}^1 \times \mathbb{P}^n$. If the linear systems are chosen sufficiently generally, then i is an embedding.

The restriction to C of a line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(a, b)$ is $\mathcal{O}_C((a + b)D + bP)$. So the canonical bundle $\omega_C \cong \mathcal{O}_C(2P)$ is the restriction of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(-2, 2)$. If the class of $D - 4P$ in $\mathrm{Pic}^0(C)$ is not torsion, then $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(-2, 2)$ is the only line bundle on $\mathbb{P}^1 \times \mathbb{P}^n$ whose restriction is ω_C. Hence the subcanonical curve $C \subset \mathbb{P}^1 \times \mathbb{P}^n$ will definitely fail such structure theorems as the Serre construction or Theorem 6.1 if the lifting condition D of Definition 0.1 fails for the isomorphism $\eta : \omega_C \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(-2, 2)|_C$.

By (1) this failure is equivalent to the nonvanishing of the composite map

$$H^1(\mathbb{P}^1 \times \mathbb{P}^n, \mathcal{O}(-2, 2)) \xrightarrow{\text{rest}} H^1(C, \mathcal{O}(-2, 2)|_C) \underset{\cong}{\xrightarrow{\eta}} H^1(C, \omega_C) \underset{\cong}{\xrightarrow{\text{tr}}} k. \tag{33}$$

Now the image of $g : C \to \mathbb{P}^2$ is a singular quintic plane curve. If we resolve the singularities, then g factors as an embedding followed by the blowdown

$C \hookrightarrow \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$. The composite map of (33) now factors through the diagram

$$\begin{tikzcd} H^1(\mathbb{P}^1\times\mathbb{P}^n, \mathcal{O}(-2,2)) \arrow[r, two heads] & H^1(\mathbb{P}^1\times\mathbb{P}^2, \mathcal{O}(-2,2)) \arrow[d, "\alpha"'] & H^1(\mathbb{P}^1\times\widetilde{\mathbb{P}}^2, \mathcal{O}(-2,2)) \cong k^6 \arrow[l, "\cong"'] \arrow[d, "\beta"] \\ & k \cong H^1(C,\omega_C) & H^1(\mathbb{P}^1\times C, \mathcal{O}(-2,2(D+P))) \cong k^9 \arrow[l, "\gamma"'] \end{tikzcd}$$

The lifting condition D fails if and only if α is surjective, hence if and only if $\operatorname{im}(\beta) \not\subset \ker(\gamma)$.

Now γ is part of the long exact sequence of cohomology for

$$0 \to \mathcal{O}_{\mathbb{P}^1\times C}(-3, D+2P) \to \mathcal{O}_{\mathbb{P}^1\times C}(-2, 2D+2P) \to \omega_C \to 0.$$

So γ is surjective, and $\ker(\gamma) \subset k^9$ is a hyperplane.

The map $g : C \to \mathbb{P}^2$ is defined using a three-dimensional subspace $U_3 \subset V_4 := H^0(C, \mathcal{O}_C(D+P))$. The complete linear system embeds $C \hookrightarrow \mathbb{P}^3$ as a curve of degree 5 and genus 2 contained in a unique quadric surface Q. Then β is the natural map from $S^2U_3 \cong k^6$ to $S^2V_4/\langle Q\rangle \cong k^9$. Now we have a range of choices for the subspace $U_3 \subset V_4$ which vary in a Zariski open subset of $\mathbb{P}^3 = \mathbb{P}(V_4^*)$. Hence we have a family of possible subspaces $S^2U_3 \subset S^2V_4$ whose different members are not all contained in any fixed hyperplane of S^2V_4. So if we choose a general $U_3 \subset V_4$, then $S^2U_3 = \operatorname{im}(\beta)$ is not contained in the hyperplane $\ker(\gamma) \subset S^2V_4/\langle Q\rangle$. In that case, $C \subset \mathbb{P}^1 \times \mathbb{P}^n$ is a subcanonical curve which fails the lifting condition D.

Singular points. Examples can easily be given of subcanonical subschemes $Z \subset X$ which are not covered by our construction because the finite projective dimension condition C of Definition 0.1 breaks down. This may happen at the same time that the lifting condition D breaks down, or it may happen independently. If D holds but C breaks down, $\mathcal{O}_Z$ will still have resolutions fitting into diagrams such as (19) of Theorem 4.1, except that $\mathcal{E}$ or $\mathcal{F}$ will not be locally free.

For instance if $X \subset \mathbb{P}^4$ is a singular hypersurface of degree d, and $P \in X$ is a singular point, then P is indeed subcanonical, but condition C fails because $\mathcal{O}_P$ is of infinite local projective dimension over $\mathcal{O}_X$. There exist isomorphisms $\eta : \mathcal{O}_P \cong \mathcal{E}xt^3_{\mathcal{O}_X}(\mathcal{O}_P, \mathcal{O}_X(\ell))$ for all $\ell \in \mathbb{Z}$, but these satisfy condition D if and only if $\ell \geq d-4$.

Similarly, if D is a line in $\mathbb{P}^5$, and $Y \subset \mathbb{P}^5$ a hypersurface containing D which is singular in at least one point of D, then condition C fails for $D \subset Y$, but all the other conditions hold (since $H^3(Y, \omega_Y(2)) = 0$). So although $D \subset Y$ may be obtained as a degeneracy locus of a pair of Lagrangian subsheaves of a twisted orthogonal bundle on Y, at least one of the Lagrangian subsheaves is not locally free.

A nonseparated example. We now give an example where there is no real choice about the η (because $H^0(\omega_Z) = k$ and there are no twists), where conditions A-C hold, but where condition D fails. The real reason for the failure in this example is that we are doing something silly on a nonseparated

scheme. But the interesting thing is that the cohomological obstruction D is able to detect our misbehavior.

Let X be the nonseparated scheme consisting of two copies $\mathbb{A}^3$ glued together along $\mathbb{A}^3 - \{0\}$. In other words, X is $\mathbb{A}^3$ with the origin doubled up. Let $P' \in X$ be one of the two origins. It is a subcanonical subscheme of X of codimension 3 of finite local projective dimension, i.e. it satisfies conditions A-C of Definition 0.1. We claim that it does not satisfy condition D.

The problem is to compute the map

$$\operatorname{Ext}^3_{\mathcal{O}_X}(\mathcal{O}_{P'}, \mathcal{O}_X) \to \operatorname{Ext}^3_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = H^3(X, \mathcal{O}_X). \tag{34}$$

We will use the following notation: $U', U'' \subset X$ are the two copies of $\mathbb{A}^3$; for $\alpha = 1, 2, 3$ let $U_\alpha \subset X$ be the open locus where $x_\alpha \neq 0$; and let $U_{\alpha\beta} := U_\alpha \cap U_\beta$, and $U_{123} := U_1 \cap U_2 \cap U_3$. For any inclusion of an affine open subscheme $U \subset X$, we denote by $i_! \mathcal{O}_U$ the extension by zero of $\mathcal{O}_U$ to all of X. We will use the same letter $i_!$ whatever the U.

Then $\mathcal{O}_X$ and $\mathcal{O}_{P'}$ have resolutions of the form

$$\begin{array}{ccccccccccc}
0 & \to & i_!\mathcal{O}_{U_{123}} & \to & \bigoplus_{\alpha<\beta} i_!\mathcal{O}_{U_{\alpha\beta}} & \to & \bigoplus_{\alpha} i_!\mathcal{O}_{U_\alpha} & \to & i_!\mathcal{O}_{U'} \oplus i_!\mathcal{O}_{U''} & \twoheadrightarrow & \mathcal{O}_X \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & i_!\mathcal{O}_{U'} & \to & i_!\mathcal{O}_{U'}^{\oplus 3} & \to & i_!\mathcal{O}_{U'}^{\oplus 3} & \to & i_!\mathcal{O}_{U'} & \twoheadrightarrow & \mathcal{O}_{P'}
\end{array}$$

The horizontal maps in the first row are more or less taken from a Čech resolution, while those from the second row are from a Koszul resolution. The vertical maps are, from left to right,

$$\left(\tfrac{1}{x_1x_2x_3}\right), \quad \begin{pmatrix} \frac{1}{x_1x_2} & 0 & 0 \\ 0 & \frac{1}{x_1x_3} & 0 \\ 0 & 0 & \frac{1}{x_2x_3} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{x_1} & 0 & 0 \\ 0 & \frac{1}{x_2} & 0 \\ 0 & 0 & \frac{1}{x_3} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

If we apply $\operatorname{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X)$ to the resolutions, we get complexes which compute the $\operatorname{Ext}^p_{\mathcal{O}_X}(\mathcal{O}_{P'}, \mathcal{O}_X)$ and the $H^p(X, \mathcal{O}_X)$. (This is because $\mathcal{O}_X$ is quasi-coherent, and the $i_!\mathcal{O}_U$ are extensions by zero of locally free sheaves on affine open subschemes.) Writing $R := k[x_1, x_2, x_3]$, these complexes are

$$\begin{array}{ccccccccccc}
0 & \to & R & \to & R^{\oplus 3} & \to & R^{\oplus 3} & \to & R & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & R \oplus R & \to & \bigoplus_{\alpha} R[x_\alpha^{-1}] & \to & \bigoplus_{\alpha<\beta} R[x_\alpha^{-1}x_\beta^{-1}] & \to & R[x_1^{-1}x_2^{-1}x_3^{-1}] & \to & 0
\end{array}$$

with Koszul and Čech horizontal arrows. The vertical arrows are as before, but transposed and in the reverse order.

The map (34) which we wish to compute may now be identified as $k \hookrightarrow H^3_{\mathfrak{m}}(R)$. This map sends a nonzero η to a nonzero multiple of socle element $x_1^{-1}x_2^{-1}x_3^{-1}$ of $H^3_{\mathfrak{m}}(R)$. So $P' \subset X$ fails condition D of the definition of a strongly subcanonical subscheme.

Actually, this example has additional pathologies which prevent $\mathcal{O}_{P'}$ from having a locally free resolution, whether symmetric or otherwise. For $\mathcal{O}_{P'}$ does not even have a locally free presentation. Indeed, for reasons of depth, any map $\mathcal{E} \to \mathcal{F}$ between locally free sheaves on X is determined by what happens outside the two origins. So the cokernel of such a map has the same fiber at the two origins. Therefore the cokernel cannot be $\mathcal{O}_{P'}$.

9. Codimension one sheaves

In this section we consider the analogues of the results in the previous sections for (skew)-symmetric sheaves of codimension 1. We include necessary and sufficient conditions for such sheaves on $\mathbb{P}^N$ to have locally free resolutions which are genuinely (skew)-symmetric, similar to those in Walter [34]. We also prove that any such sheaf on a quasi-projective variety has a resolution which is (skew)-symmetric up to quasi-isomorphism, in analogy with Theorem 6.1. We finish the section with several examples.

In this section we will suppose that the characteristic is not 2, although all the theorems have variants which are valid in characteristic 2.

Symmetric sheaves of codimension 1. Suppose $\mathcal{F}$ is a coherent sheaf on a scheme X which is of finite local projective dimension and perfect of codimension 1. This means that locally $\mathcal{F}$ has free resolutions $0 \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0$ such that the dual complex $0 \to \mathcal{L}_0^* \to \mathcal{L}_1^* \to \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \to 0$ is also exact. The operation

$$\mathcal{F} \rightsquigarrow \mathcal{F}^\vee := \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$$

provides a duality on the category of such sheaves. A *symmetric sheaf of codimension* 1 is a pair $(\mathcal{F}, \alpha)$ where $\mathcal{F}$ is a sheaf as above, and $\alpha : \mathcal{F} \to \mathcal{F}^\vee(L)$ is an isomorphism which is symmetric in the sense that $\alpha = \alpha^\vee$. (Here L is some line bundle on X.) *Skew-symmetric sheaves of codimension* 1 on X are defined similarly.

Symmetric resolutions in codimension 1. Resolutions of codimension 1 symmetric sheaves on $\mathbb{P}^3$ have been studied fairly extensively by Barth [4], Casnati-Catanese [7], and Catanese [8] [9], in the context of surfaces with even sets of nodes and by Kleiman-Ulrich [23] in the context of self-linked curves. The next theorem, conjectured by Barth and Catanese, was proven by Casnati-Catanese for symmetric sheaves on $\mathbb{P}^3$ ([7] Theorem 0.3). They also remarked ([7] Remark 2.2) that essentially the same proof works for codimension 1 symmetric sheaves on any $\mathbb{P}^n$, which is true as long as one remembers to include in one's statement a parity condition analogous to that in Walter [34] Theorem 0.1. For a case where the parity condition fails, see Example 9.3 below.

Theorem 9.1 ([7] [9] with correction)**.** *Let k be an algebraically closed field of characteristic different from* 2*. Suppose that $(\mathcal{F}, \alpha)$ is a symmetric sheaf of*

codimension 1 *on* $\mathbb{P}^n_k$, *with* $\alpha : \mathcal{F} \xrightarrow{\sim} \mathcal{F}^\vee(\ell - n - 1)$. *Then* $\mathcal{F}$ *has a symmetric resolution, i.e. a locally free resolution of the form*

$$0 \to \mathcal{G} \xrightarrow{f} \mathcal{G}^*(\ell - n - 1) \to \mathcal{F} \to 0$$

with f *symmetric, if and only if the following parity condition holds*: *if* $n \equiv 1 \pmod 4$ *and* ℓ *is even, then* $\chi(\mathcal{F}(-\ell/2))$ *is also even.*

A higher-codimension generalization of this theorem is proven in our paper [14].

As the parity condition indicates, symmetric sheaves do not always possess symmetric resolutions. The following structure theorem, analogous to Theorem 0.2, shows that they do still have locally free resolutions which are symmetric up to quasi-isomorphism.

Theorem 9.2. *Let* X *be a quasiprojective scheme over a Noetherian ring and let* $\mathcal{F}$ *be a coherent sheaf. The following are equivalent:*

(a) $\mathcal{F}$ *is perfect of codimension one, and there exists a line bundle* L *on* X *and an isomorphism* $\alpha : \mathcal{F} \to \mathcal{F}^\vee(L)$ *making* $(\mathcal{F}, \alpha)$ *a symmetric sheaf.*

(b) *The sheaf* $\mathcal{F}$ *has symmetrically quasi-isomorphic locally free resolutions*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G} \arrow[r, "\psi"] \arrow[d, "\phi"'] & \mathcal{H} \arrow[r] \arrow[d, "\phi^*"] & \mathcal{F} \arrow[r] \arrow[d, "\alpha", "\cong"'] & 0 \\ 0 \arrow[r] & \mathcal{H}^*(L) \arrow[r, "\psi^*"] & \mathcal{G}^*(L) \arrow[r] & \mathcal{F}^\vee(L) \arrow[r] & 0 \end{tikzcd} \tag{35}$$

with $\phi^*\psi : \mathcal{G} \to \mathcal{G}^*(L)$ *a symmetric map and* L *a line bundle on* X.

(c) *There exists a line bundle* L *on* X *and a Lagrangian subbundle of a twisted symplectic bundle*

$$\mathcal{G} \xhookrightarrow{\binom{\psi}{\phi}} \mathcal{H} \oplus \mathcal{H}^*(L)$$

such that

$$0 \longrightarrow \mathcal{G} \xrightarrow{\psi} \mathcal{H} \longrightarrow \mathcal{F} \longrightarrow 0$$

is a resolution of $\mathcal{F}$ *fitting into a commutative diagram as in* (35).

Proof. The only delicate part is (a) $\Rightarrow$ (b). Because $\mathcal{F}$ is locally Cohen-Macaulay of codimension 1, it has a locally free resolution $0 \to \mathcal{P}_1 \to \mathcal{P}_0 \to \mathcal{F} \to 0$. The symmetric isomorphism α corresponds to a morphism in the derived category

$$\begin{tikzcd} S^2(\mathcal{P}_\bullet): \arrow[d, dotted, "\alpha"] & 0 \arrow[r] & \Lambda^2\mathcal{P}_1 \arrow[r] \arrow[d, dotted] & \mathcal{P}_1 \otimes \mathcal{P}_0 \arrow[r] \arrow[d, dotted] & S^2\mathcal{P}_0 \arrow[r] \arrow[d, dotted] & 0 \\ L[1]: & 0 \arrow[r] & 0 \arrow[r] & L \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

This α is a member of the hyperext $\mathbb{E}\mathrm{xt}^1_{\mathcal{O}_X}(S^2(\mathcal{P}_\bullet), L)$, which in turn is the abutment of the hyperext spectral sequence

$$E_1^{pq} = \mathrm{Ext}^q_{\mathcal{O}_X}((S^2(\mathcal{P}_\bullet))_p, L) \Longrightarrow \mathbb{E}\mathrm{xt}^{p+q}_{\mathcal{O}_X}(S^2(\mathcal{P}_\bullet), L).$$

The differentials d_1 define complexes (indexed by $p = 0, 1, 2$)

$$0 \to \mathrm{Ext}^q_{\mathcal{O}_X}(S^2\mathcal{P}_0, L) \xrightarrow{d_1} \mathrm{Ext}^q_{\mathcal{O}_X}(\mathcal{P}_1 \otimes \mathcal{P}_0, L) \xrightarrow{d_1} \mathrm{Ext}^q_{\mathcal{O}_X}(\Lambda^2\mathcal{P}_1, L) \to 0$$

whose cohomology groups are the E_2^{pq}. In particular, E_2^{10} is the space of homotopy classes of chain maps $S^2(\mathcal{P}_\bullet) \to L[1]$. Hence α will be the class of an honest chain map if and only if it comes from E_2^{10}. However, according to the 5-term exact sequence

$$0 \to E_2^{10} \to \mathbb{E}\mathrm{xt}^1_{\mathcal{O}_X}(S^2(\mathcal{P}_\bullet), L) \to E_2^{01} \to \cdots,$$

the obstruction lies in $E_2^{01} \subset \mathrm{Ext}^1_{\mathcal{O}_X}(S^2\mathcal{P}_0, L)$. As in the proof of Theorem 6.1, this obstruction may be nonzero, but it can be killed by pulling back along a suitable epimorphism $\mathcal{H} \twoheadrightarrow \mathcal{P}_0$ (cf. Lemma 6.2). The proof may now be completed with arguments taken from the proof of Theorem 6.1. $\square$

We now use this theorem to construct an example of a symmetric codimension 1 sheaf on $\mathbb{P}^5$ for which the parity condition of Theorem 9.1 fails. The construction is similar to the main examples of [13].

Example 9.3. Let $V = H^0(\mathcal{O}_{\mathbb{P}^5}(1))^*$. The exterior product defines a symplectic form on the 20-dimensional vector space $\Lambda^3 V$, which makes the trivial bundle $\Lambda^3 V \otimes \mathcal{O}_{\mathbb{P}^5}$ into a symplectic bundle. If $W, W^* \subset \Lambda^3 V$ are general Lagrangian subspaces, then we can identify the symplectic vector space $\Lambda^3 V$ with hyperbolic symplectic vector space $W \oplus W^*$. Moreover, one can see that $\Omega^3_{\mathbb{P}^5}(3)$ is a Lagrangian subbundle of $\Lambda^3 V \otimes \mathcal{O}_{\mathbb{P}^5}$ (cf. [13], §5). The construction of Theorem 9.2 then produces a symmetric codimension 1 sheaf $\mathcal{F}$ on $\mathbb{P}^5$ with resolutions

$$\begin{tikzcd}
0 \arrow[r] & \Omega^3_{\mathbb{P}^5}(3) \arrow[r, "\psi"] \arrow[d, "\phi"'] & W \otimes \mathcal{O}_{\mathbb{P}^5} \arrow[r] \arrow[d, "\phi^*"] & \mathcal{F} \arrow[r] \arrow[d, "\alpha", "\cong"'] & 0 \\
0 \arrow[r] & W^* \otimes \mathcal{O}_{\mathbb{P}^5} \arrow[r, "\psi^*"] & \Omega^2_{\mathbb{P}^5}(3) \arrow[r] & \mathcal{F}^\vee \arrow[r] & 0.
\end{tikzcd} \tag{36}$$

The sheaf $\mathcal{F}$ fails the parity condition of Theorem 9.1 because $\ell = 6$ and $\chi(\mathcal{F}(-3)) = 1$.

The geometry of the sheaf $\mathcal{F}$ is best explained using the degeneracy loci of the Lagrangian subbundles $\Omega^3_{\mathbb{P}^5}(3)$ and $W^* \otimes \mathcal{O}_{\mathbb{P}^5}$ of $\Lambda^3 V \otimes \mathcal{O}_{\mathbb{P}^5}$:

$$D_i := \{x \in \mathbb{P}^5 \mid \dim \left[\Omega^3_{\mathbb{P}^5}(3)(x) \cap W^*\right] \geq i\}.$$

The sheaf $\mathcal{F}$ is supported on the sextic fourfold D_1. If W^* is general, then D_1 is smooth (cf. [13], Theorem 2.1 and the discussion following it) except along the surface D_2 where it has A_1 singularities with local equations $x_1^2 + x_2^2 + x_3^2 = 0$. The surface D_2 is of degree 40 according to the formulas of Fulton-Pragacz [17] (6.7).

Now choose a general 9-dimensional subspace U of the 10-dimensional space W^*. Then the composite map $U \otimes \mathcal{O}_{\mathbb{P}^5} \hookrightarrow \Lambda^3 V \otimes \mathcal{O}_{\mathbb{P}^5} \twoheadrightarrow \Omega^2_{\mathbb{P}^5}(3)$ degenerates in codimension 2 along a threefold Y of degree 18. Since

$$Y = \{x \in \mathbb{P}^5 \mid \dim \left[\Omega^3_{\mathbb{P}^5}(3)(x) \cap U\right] \geq 1\},$$

we have $D_2 \subset Y \subset D_1$. Moreover, $\mathcal{F} \cong \mathcal{I}_{Y/D_1}(6)$. In addition, Y is self-linked by the complete intersection of D_1 and of another sextic hypersurface corresponding to another Lagrangian subspace of $\Lambda^3 V$ containing U.

Skew-symmetric sheaves of codimension one. Analogues of Theorems 9.1 and 9.2 hold for skew-symmetric sheaves of codimension 1. The only significant change is in the parity condition of Theorem 9.1, which in the skew-symmetric case has the form: "if $n \equiv 3 \pmod 4$ and ℓ is even, then $\chi(\mathcal{F}(-\ell/2))$ is also even." We leave the exact formulation of these results to the reader.

If $S \subset \mathbb{P}^3$ is a smooth surface of degree d, then its cotangent bundle Ω_S is a skew-symmetric sheaf of codimension 1 on $\mathbb{P}^3$ with twist $\ell = 0$. Since $\chi(\Omega_S) = -h^{11}(S) \equiv d \pmod 2$, this skew-symmetric sheaf fails the parity condition when d is odd.

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Department of Mathematics, University of California, Berkeley, Berkeley CA 94720
E-mail address: de@msri.org
URL: http://www.msri.org/people/staff/de/

Department of Mathematics, Columbia University, New York, NY 10027
E-mail address: psorin@math.columbia.edu
URL: http://www.math.columbia.edu/~psorin/

Laboratoire J.-A. Dieudonné (UMR CNRS 6621), Université de Nice – Sophia Antipolis, 06108 Nice Cedex 02, France
E-mail address: walter@math.unice.fr
URL: http://math1.unice.fr/~walter/